

# THE HEAT KERNEL ON AN ASYMPTOTICALLY CONIC MANIFOLD

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**ABSTRACT.** In this paper, we investigate the long-time structure of the heat kernel on a Riemannian manifold  $M$  which is asymptotically conic near infinity. Using geometric microlocal analysis and building on results of Guillarmou and Hassell in [GH1], we give a complete description of the asymptotic structure of the heat kernel in all spatial and temporal regimes. We apply this structure to define and investigate a renormalized zeta function and determinant of the Laplacian on  $M$ .

## 1. INTRODUCTION

We study the heat kernel on asymptotically conic manifolds. Asymptotically conic manifolds should be thought of as those complete manifolds which are approximately conic near infinity. More specifically, we have the following definition [GH1]:

**Definition.** Let  $(M, g)$  be a complete Riemannian manifold without boundary of dimension  $n$ , and let  $\bar{M}$  be the usual radial compactification of  $M$ . Let  $(N, h_0)$  be a closed Riemannian manifold of dimension  $n - 1$ . We say that  $(M, g)$  is asymptotically conic with cross-section  $(N, h_0)$  if in a neighborhood of  $\partial\bar{M}$ ,  $\bar{M}$  is isometric to  $[0, \delta)_x \times N_y$  with the metric

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}. \quad (1)$$

Here  $x$  is a smooth function on  $\bar{M}$  with  $x = 0$  and  $dx \neq 0$  on  $\partial\bar{M}$  (we call this a boundary defining function for  $\partial\bar{M}$ ) and a smooth family of metrics  $h(x)$  on  $N$  with  $h(0) = h_0$ .

In particular, Euclidean space  $\mathbb{R}^n$  is asymptotically conic with cross-section  $S^{n-1}$ ; we may choose  $x = r^{-1}$ . Any complete manifold which is exactly Euclidean or conic near infinity is, of course, also asymptotically conic. The condition (1) may be weakened by replacing  $h(x)$  with any symmetric 2-tensor  $h'(x, y)$  which restricts to a metric  $h_0(x)$  on the boundary at  $x = 0$ ; an observation of Melrose and Wunsch [MW] shows that these conditions are in fact equivalent.

Asymptotically conic manifolds are a relatively well-behaved class of manifolds, and as such the theory of the heat equation is relatively advanced. In particular, it is easy to see from (1) that all sectional curvatures of  $(M, g)$  approach zero as  $x$  goes to zero, and thus that  $(M, g)$  has bounded sectional curvature. For complete manifolds of bounded sectional curvature, a classical theorem of Cheng-Li-Yau [CLY] gives the following Gaussian upper bound for the heat kernel:

**Theorem 1.** [CLY] *There are nonzero constants  $C_1$  and  $C_2$  such that the heat kernel on  $M$ , denoted  $H^M(t, z, z')$ , satisfies*

$$H^M(t, z, z') \leq \frac{C_1}{t^{n/2}} e^{-\frac{|z-z'|^2}{C_2 t}}. \quad (2)$$

However, for many applications to spectral theory, one needs finer information about the structure of the heat kernel. The example we have in mind is the definition of the zeta function. Recall that if  $M$  is compact, the zeta function is defined for  $\Re s > n/2$  by:

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr} H^M(t) - 1) t^{s-1} dt. \quad (3)$$

The zeta function has a well-known meromorphic continuation to all of  $\mathbb{C}$  with a regular value at  $s = 0$ ; the key is that the trace of the heat kernel has a short-time asymptotic expansion, which, along with the long-time exponential decay, enables us to write down an explicit meromorphic continuation [R]. The *determinant of the Laplacian* is then given by  $\exp(-\zeta'_M(0))$ ; the determinant plays a key role in many problems in spectral theory, including the isospectral compactness results of Osgood, Phillips, and Sarnak [OPS1, OPS2, OPS3].

We would like to define such a zeta function and determinant when  $M$  is asymptotically conic, with an eye towards applying these concepts to the spectral and scattering theory of asymptotically conic manifolds. There are several obstacles. First, the heat kernel is no longer trace class, so  $\text{Tr} H^M(t)$  does not make sense. Instead, we define the *renormalized heat trace*  ${}^R\text{Tr} H^M(t)$  to be the finite part at  $\delta = 0$  of the divergent asymptotic expansion in  $\delta$  of

$$\int_{x \geq \delta} H^M(t, x, y, x, y) x^{-n-1} dx dy. \quad (4)$$

Details, including the existence of this divergent asymptotic expansion, may be found in Section 3. We then formally define the *renormalized zeta function*:

$${}^R\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty {}^R\text{Tr} H^M(t) t^{s-1} dt. \quad (5)$$

However, to make sense of this definition and obtain a meromorphic continuation, we still need to understand the behavior of the renormalized trace - and hence of the heat kernel itself - as  $t \rightarrow 0$  and  $t \rightarrow \infty$ . In particular, we need asymptotics in both the short and long time regimes.

The short-time behavior of the heat kernel on an asymptotically conic manifold is relatively well-understood. Short-time heat kernels may be analyzed using techniques from semiclassical analysis. In this approach, the goal is to develop a 'semiclassical functional calculus' containing the heat kernel, modeled on standard semiclassical techniques as developed, for example, in [DS]. The key functional calculus for this purpose, at least in the asymptotically Euclidean setting, is the Weyl calculus of Hormander [Ho].

An alternate approach, and one more suited to analysis of the renormalized trace and determinant, is to use geometric microlocal analysis to first construct the heat kernel and then analyze its fine structure. The techniques of geometric microlocal analysis were first developed by Melrose and Mendoza to study elliptic PDE on manifolds with asymptotically cylindrical ends [MeMe]. They have been extended by many other mathematicians and play a key role in the modern analysis of linear PDE on singular and non-compact spaces. In particular, in [Me3], Melrose discusses some aspects of spectral and scattering theory on asymptotically conic manifolds. Albin, in [Alb], uses these methods to investigate the short-time heat kernel on a variety of complete spaces, including asymptotically conic manifolds. His work can be used to obtain the fine structure that we need for the short-time heat kernel.

The long-time problem is trickier: in the asymptotically conic setting, we no longer have exponential decay of the heat kernel as  $t \rightarrow \infty$ . Indeed, from the structure of the Euclidean

heat kernel and (2), we expect that the leading-order behavior of  $H^M(t, z, z')$  as  $t \rightarrow \infty$  will be  $Ct^{-n/2}$ , and the leading-order behavior of the renormalized heat trace may be even worse. This lack of decay means that the zeta function may not be well-defined a priori for any  $s$ . We may split (5) into two integrals by breaking it up at  $t = 1$ , but there is no obvious reason for the integral from  $t = 1$  to  $\infty$  to have a meromorphic continuation to all of  $\mathbb{C}$ . In order to obtain such a meromorphic continuation, we need an asymptotic expansion for the heat kernel as  $t \rightarrow \infty$ . Moreover, we must understand how this expansion interacts with the heat trace renormalization.

**1.1. Main results.** We solve this problem by using the methods of geometric microlocal analysis to obtain a complete description of the asymptotic structure of the heat kernel on  $M$  in all spatial and temporal regimes. Our key result is the following:

**Theorem 2.** *Let  $M$  be asymptotically conic. For any  $n \geq 2$ , and for any fixed time  $T > 0$ , the heat kernel on  $M$  is polyhomogeneous conormal on  $M_{w,sc}^2$  for  $t > T$ , where  $w = t^{-1/2}$ . The leading orders at the boundary hypersurfaces are at least 0 at  $sc$  and  $n$  at each of  $bf_0$ ,  $rb_0$ ,  $lb_0$ , and  $zf$ , with infinite-order decay at  $lb$ ,  $rb$ , and  $bf$ .*

This theorem gives a complete description of the asymptotic structure of the heat kernel for long time; previously, only estimates such as Theorem [CLY] were known. The space  $M_{w,sc}^2$  is a manifold with corners which is a blow-up of  $(w, x, y, x', y')$ -space; we give its explicit definition in Section 2, and also briefly discuss blow-ups and polyhomogeneous conormal functions. These concepts were originally introduced in [Me] and [Me2], and a good introduction may be found in [Gr].

The analogous structure for the short-time heat kernel is well-understood:

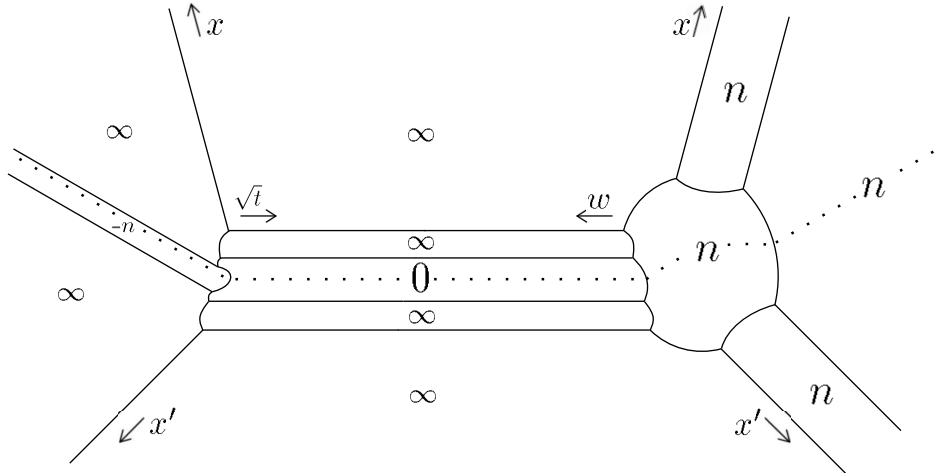
**Theorem 3.** *For  $t < 1$ , the heat kernel  $H^M(t, x, y, x', y')$  is polyhomogeneous conormal on  $[M_{sc}^2(x, y, x', y') \times [0, 1]_{\sqrt{t}}; \{\sqrt{t} = 0, x = x', y = y'\}]$ .*

*Moreover, there is infinite-order decay at all faces except the scattering front face  $sc$  and the face  $F$  obtained by the final blowup.*

Theorem 3 follows immediately from the work of Albin [Alb]; however, the precise statement above does not appear in the literature. Therefore, in the Appendix, we give a simple proof using the machinery developed in [Alb]. Combining Theorem 2 with Theorem 3 gives a complete geometric-microlocal description of the structure of the heat kernel. This structure is illustrated in Figure 1; the short-time structure is the left-hand side of the diagram, and the long-time structure is the right-hand side. We also indicate the leading order of the heat kernel at each of the boundary hypersurfaces, in terms of  $\sqrt{t}$  at  $t = 0$ ,  $w = t^{-1/2}$  at  $t = \infty$ , and  $x$  or  $x'$  respectively at all the finite-time boundaries.

As an example of how polyhomogeneous structure may be used to read off the behavior in all asymptotic regimes, let  $a$  be a parameter, and fix  $(y, x', y')$ ; consider the heat kernel  $H^Z(a^2, a^{-1}, y, x', y')$  as  $a$  approaches infinity. In the compactified space in Figure 1, as  $a$  approaches infinity, the arguments approach a point in the center of the face  $lb_0$ , where the leading order of the heat kernel is  $n$ . Since  $w = a^{-1}$ , and  $w$  is a boundary defining function for  $lb_0$ , we conclude that  $H^Z(a^2, a^{-1}, y, x', y')$  has a polyhomogeneous asymptotic expansion in  $a^{-1}$  as  $a \rightarrow \infty$ , with leading term  $C_n a^{-n}$ . A similar analysis may be performed in any asymptotic regime.

As an application, Theorems 2 and 3 give us precisely the polyhomogeneous structure we need to define and investigate the renormalized zeta function on  $M$ :

FIGURE 1. Asymptotic structure of the heat kernel on  $M$ .

**Theorem 4.** *Let  $M$  be asymptotically conic. The renormalized zeta function, defined formally by (5), is well-defined and has a meromorphic continuation to all of  $\mathbb{C}$ .*

We may then define the *renormalized determinant* of the Laplacian on  $M$  by

$$\log({}^R\det \Delta_M) = -{}^R\zeta'_M(0),$$

where  ${}^R\zeta'_M(0)$  is the coefficient of  $s$  in the Laurent series for  ${}^R\zeta_M(s)$  around  $s = 0$ .

In a companion paper [S2], we use these theorems to analyze the behavior of the determinant of the Laplacian on a family of manifolds degenerating to a manifold with conical singularities. We expect that this work will have applications to spectral theory and to index theory on singular spaces, including the study of the Cheeger-Müller theorem on manifolds with conical singularities.

**1.2. Outline of the proofs.** The usual geometric-microlocal approach to the fine structure of the heat kernel is a direct parametrix construction, which involves the construction of an initial approximation to the heat kernel and then the removal of the error via a Neumann series argument. This is the method adopted in [Alb]. However, parametrix constructions are not well-suited for analysis of the long-time heat kernel; the problem is global rather than local. In order to obtain the asymptotic structure of the heat kernel at long time, we instead take an indirect approach. Recall that the functional calculus shows that the heat kernel and the resolvent are related by

$$H^M(t) = \int_{\Gamma} e^{\lambda t} (\Delta_M + \lambda)^{-1} d\lambda, \quad (6)$$

where  $\Gamma$  is a contour around the spectrum. In a series of papers [GH1, GH2], Guillarmou and Hassell have analyzed the asymptotic structure of the resolvent  $(\Delta_M + \lambda)^{-1}$  at low energy, again giving a complete description in all regimes. They have shown:

**Theorem 5.** [GH1] *Suppose that  $M$  is an asymptotically conic manifold of dimension  $n \geq 3$ . Then the Schwarz kernel of  $(\Delta_M + e^{i\theta}k^2)^{-1}$  is polyhomogeneous conormal on  $M_{k,sc}^2$  for each  $\theta \in (-\pi, \pi)$ , with a conormal singularity at the spatial diagonal and all coefficients smoothly depending on  $\theta$ . It decays to infinite order at the faces  $lb$ ,  $rb$ , and  $bf$ , with leading orders at  $sc$ ,  $bf_0$ ,  $rb_0$ ,  $lb_0$  and  $zf$  given by  $0$ ,  $n-2$ ,  $n-2$ ,  $n-2$ , and  $0$  respectively.*

Note that Guillarmou and Hassell require  $n \geq 3$ . In Section 4, we adapt the methods of [GH1] to extend Theorem 5 to the two-dimensional case:

**Theorem 6.** *Theorem 5 also holds when  $n = 2$ ; all the leading orders are the same, except that we have logarithmic growth instead of order 0 at  $zf$ .*

In Section 2, we use geometric microlocal analysis, in particular Melrose's pushforward theorem, to prove Theorem 2. The key is to push the structure of Theorems 5 and 6 through the contour integral (6). The main technical theorem, which we state in slightly greater generality, is the following:

**Theorem 7.** *Let  $M$  be an asymptotically conic manifold, and let  $E$  be a vector bundle over  $M$ . Let  $A : C^\infty(E) \rightarrow C^{-\infty}(E)$  be a pseudodifferential operator with the following properties:*

- a)  $\sigma(A) \subset [0, \infty]$ ;*
- b) The Schwarz kernel of the resolvent  $(A + e^{i\theta}k^2)^{-1}$  is polyhomogeneous conormal on  $M_{k,sc}^2$  for each  $\theta \in (-\pi, \pi)$ , with a conormal singularity at the spatial diagonal and all coefficients smoothly depending on  $\theta$ . Moreover, it decays to infinite order at the faces  $lb$ ,  $rb$ , and  $bf$ , with index sets at  $sc$ ,  $bf_0$ ,  $rb_0$ ,  $lb_0$  and  $zf$  given by  $R_{sc}$ ,  $R_{bf_0}$ ,  $R_{rb_0}$ ,  $R_{lb_0}$ , and  $R_{zf}$  respectively;*
- c) The resolvent at high energy is polynomially bounded in  $k$  along with all derivatives. In particular, there exist  $\varphi \in (\pi/2, \pi)$  and  $N \in \mathbb{N}$  with the following property: for any b-differential operator  $Q$  in  $k^{-1}$  and any scattering differential operator  $P$  on  $M$ , there is a constant  $C$  such that*

$$|PQ(A + e^{i\theta}k^2)^{-1}(x, y, x', y')| \leq Ck^N$$

*for all  $\theta \in [-\varphi, \varphi]$  and all  $k \geq 1$ .*

*Then for  $t$  greater than any fixed  $T > 0$ , the kernel of  $e^{-tA}$  is polyhomogeneous conormal on  $M_{w,sc}^2$ , where  $w = t^{-1/2}$ . It decays to infinite order at  $lb$ ,  $rb$ , and  $bf$ , and has index sets at  $sc$ ,  $bf_0$ ,  $rb_0$ ,  $lb_0$ , and  $zf$  which are subsets of  $R_{sc}$ ,  $R_{bf_0} + 2$ ,  $R_{rb_0} + 2$ ,  $R_{lb_0} + 2$ , and  $R_{zf} + 2$  respectively.*

Once we have proven Theorem 7, Theorem 2 is an almost immediate consequence, though there is a slight twist involving the leading orders.

In section 3, we use Theorem 2 and some additional geometric microlocal techniques to analyze the renormalized heat trace and prove Theorem 4. We also analyze the renormalized zeta function and determinant in the special case where  $M$  is exactly conic (or Euclidean) outside a compact set. In section 4, we extend Guillarmou and Hassell's work in [GH1] to prove Theorem 6. Finally, in the Appendix, we use the framework in [Alb] to prove Theorem 3.

**1.3. Acknowledgements.** This work comprises the first part of my Stanford Ph.D. thesis [S]. I owe a great deal to my advisor, Rafe Mazzeo, who introduced me to this problem and shared tremendous advice and support. It would be impossible to thank everyone who contributed to this project, but I would like to particularly thank Pierre Albin, Colin Guillarmou, Andrew Hassell, and Andras Vasy for providing interesting and helpful ideas. I am grateful to Gilles Carron and the Université de Nantes for their generous hospitality during the fall of 2010, during which a significant portion of this work was completed. Finally, I would like to thank the ARCS foundation for financial support in the 2011-2012 academic year.

## 2. FROM RESOLVENT TO HEAT KERNEL

The goal of this section is to prove Theorems 7 and 2.

**2.1. Preliminaries.** We first give a brief summary of the key relevant concepts in geometric microlocal analysis; again, a self-contained introduction may be found in [Gr]. A *manifold with corners* of dimension  $n$  is a topological space which is locally modeled on  $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$  for some  $k$ ; a simple example is the  $n$ -dimensional unit cube. *Blow-up* is a way of creating new manifolds with corners from old ones, and is used to resolve certain geometric singularities. The idea is to formally introduce polar coordinates around a submanifold of a manifold with corners, in order to distinguish between directions of approach to that submanifold. For example, consider the origin as a submanifold of  $\mathbb{R}_+^2$ . To blow up the origin, we introduce polar coordinates  $(r, \theta)$ , which corresponds to replacing the point  $(0, 0)$  with a quarter-circle, which corresponds to the inward-pointing spherical normal bundle of  $(0, 0) \subset \mathbb{R}_+^2$ . See [Me], [Me2], [Gr], and/or the appendix of [S] for a more detailed explanation and more general examples of blow-ups.

By Taylor's theorem, smooth functions on a manifold with corners are precisely those functions which have Taylor expansions at each boundary hypersurface and joint Taylor expansions at every corner. *Polyhomogeneous conormal distributions*, which we abbreviate as phg or phg conormal, are a generalization of smooth functions. In particular, if we let  $x$  be a boundary defining function, we allow terms of the form  $x^s(\log x)^p$  for any  $x \in \mathbb{C}$  and any  $p \in \mathbb{N}_0$  to appear in the asymptotic expansions at the boundary and in the joint expansions at the corners. The *index set* of a phg conormal distribution  $u$  at a particular boundary hypersurface  $H$  is simply the set of  $(s, p)$  which appear in the asymptotic expansion of  $u$  at  $H$ .

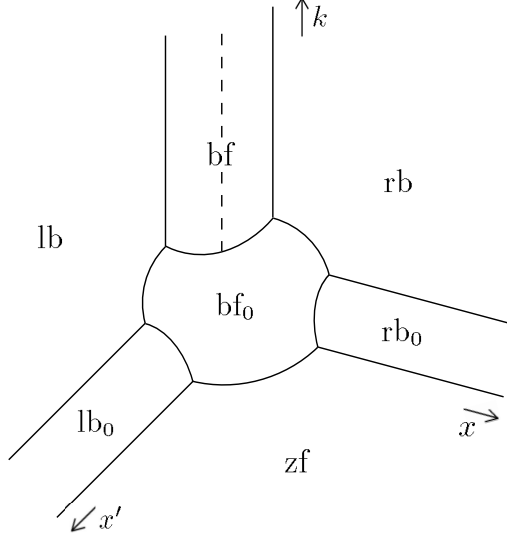
Polyhomogeneous conormal functions are well-behaved under addition and multiplication, but also under more complicated operations, namely pull-back and push-forward. To discuss these, we first need to discuss properties of a map  $f : W \rightarrow Z$  between manifolds with corners. Roughly, we say that  $f$  is a *b-map* if it is smooth up to the boundary and product-type near the boundary in terms of the local coordinate models (see [Gr] for a precise definition). If additionally  $f$  does not map any boundary hypersurface of  $W$  into a corner of  $Z$ , and  $f$  is also a fibration over the interior of every boundary hypersurface, then we call  $f$  a *b-fibration*. We now have the following two theorems, due to Melrose [Me, Me2], which will be critical in the analysis to follow:

**Proposition 8** (Melrose's Pull-Back and Push-Forward Theorems). *Let  $f : W \rightarrow Z$  be a smooth map of manifolds with corners.*

*a) If  $f$  is a b-map and  $u$  is phg conormal on  $Z$ , then  $f^*u$  is phg conormal on  $W$ . Moreover, the index sets of  $f^*u$  may be computed explicitly from those of  $u$  and the geometry of the map  $f$ .*

*b) If  $f$  is also a b-fibration,  $v$  is phg conormal on  $W$ , and  $f_*v$  is well-defined (the pushforward is integration along the fibers, which may not converge), then  $f_*v$  is phg conormal on  $Z$ , and again the index sets may be computed explicitly.*

Finally, we need to consider distributions which have pseudodifferential-type *conormal singularities* at submanifolds in the interior of a manifold with corners.

FIGURE 2. First blowup in creation of  $M_{k,sc}^2$ .

**Definition.** [Gr] Let  $y = (x_1, \dots, x_k)$  and  $z = (x_{k+1}, \dots, x_n)$ . A distribution  $u$  on  $\mathbb{R}^n$  has a conormal singularity at  $N$  of order  $m$  if it can be written

$$u(y, z) = \int_{\mathbb{R}^{n-k}} e^{iz \cdot \xi} a(y, \xi) d\xi,$$

where  $a$  is a classical symbol; that is,  $a$  has asymptotics as  $|\xi| \rightarrow \infty$

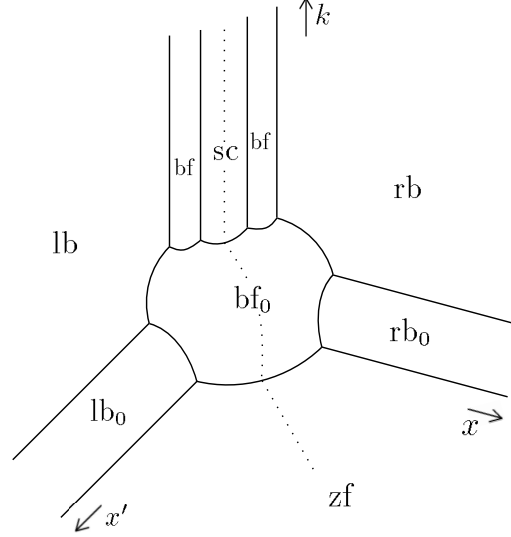
$$a(y, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(y, \frac{\xi}{|\xi|}) |\xi|^{m-j},$$

with each coefficient  $a_{m-j}$  smooth in  $y$  and  $\frac{\xi}{|\xi|}$ .

This definition may be extended, by using the local coordinate models, to define distributions with a conormal singularity at any  $p$ -submanifold of a manifold with corners; a  $p$ -submanifold is a subset which, in each local coordinate chart, may be identified with a coordinate submanifold. Variants of the pullback and pushforward theorems also hold for polyhomogeneous conormal distributions with interior conormal singularities [Me2, EMM].

**2.2. The space  $M_{k,sc}^2$ .** We now introduce the space  $M_{k,sc}^2$ , which first appears in an unpublished note of Melrose and Sa Barreto [MSB], and later in [GH1]. To construct  $M_{k,sc}^2$ , we begin with the space  $M_k^2 = [0, 1] \times \bar{M} \times \bar{M}$ ; coordinates on this space are  $(k, x, y, x', y')$ . There are three boundary hypersurfaces:  $\{k = 0\}$ , which we call zf,  $\{x = 0\}$ , which we call lb, and  $\{x' = 0\}$ , which we call rb.

First we blow up the corner  $\{x = 0, x' = 0, k = 0\}$ , which corresponds to the introduction of polar coordinates near that corner; we call the front face of this blowup  $bf_0$ . We then blow up three codimension-2 submanifolds: we blow up  $\{x = 0, x' = 0\}$  and call the new face bf, we blow up  $\{x = 0, k = 0\}$  and call the resulting face  $lb_0$ , and we blow up  $\{x' = 0, k = 0\}$  and call the resulting face  $rb_0$ . The resulting manifold with corners is shown in Figure 2, with  $y$  and  $y'$  suppressed. Using the definition of a "b-stretched product" from the work

FIGURE 3. The space  $M_{k,sc}^2$ .

of Melrose-Singer [MS], it is precisely  $X_b^3(x, x', k) \times N_y \times N_{y'}$ . We use these b-stretched products  $X_b^n$  from [MS] throughout the arguments.

Finally, consider the intersection of the closure of the interior diagonal  $\{x = x' > 0, y = y'\}$  with the face bf. In coordinates, this is  $\{(x/x') = 1, y = y', (x/k) = 0\}$ , and it is marked with a dotted line in Figure 2. We blow this up to create a new boundary hypersurface, which we call sc (for “scattering”). The resulting space is  $M_{k,sc}^2$ , and it has eight boundary hypersurfaces, illustrated in Figure 3. The spatial diagonal  $D_{k,sc}$  is defined to be the closure in  $M_{k,sc}^2$  of the interior spatial diagonal  $\{x = x' > 0, y = y'\}$ ; its intersection with the boundary is marked with a dotted line in Figure 3.

We now describe some useful coordinate systems on  $M_{w,sc}^2$ . Near the intersection of zf, rb<sub>0</sub>, and bf<sub>0</sub>, we use the coordinates

$$(x, \sigma = \frac{x'}{x}, y, y', \kappa' = \frac{k}{x}).$$

In these coordinates,  $x$  is a boundary defining function (bdf) for bf<sub>0</sub>,  $\sigma$  is a bdf for rb<sub>0</sub>, and  $\kappa'$  is a bdf for zf. Similarly, near the intersection of zf, lb<sub>0</sub>, and bf<sub>0</sub>, we use the coordinates

$$(x', \sigma' = \frac{x}{x'}, y, y', \kappa = \frac{k}{x}).$$

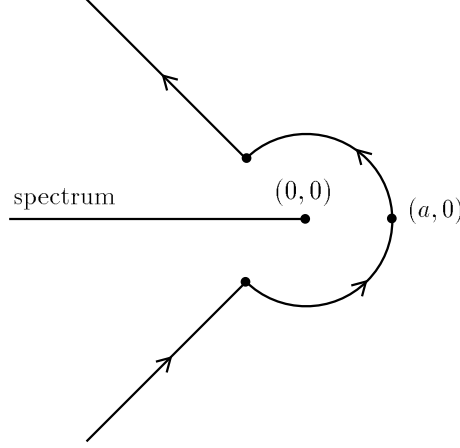
Coordinates near sc are slightly more complicated; before the final blowup, good coordinates are  $(x', \sigma', y, y', (x/k))$ . After the blowup, we use the coordinates

$$X = k(\frac{1}{x'} - \frac{1}{x}), Y = k\frac{y - y'}{x}, \lambda = \frac{x}{k}, y, k.$$

These are valid in a neighborhood of the intersection of  $D_{k,sc}$  with sc and bf<sub>0</sub>; however, they are not good coordinates as we approach bf.

**2.3. Proof of Theorem 7.** Let  $A$  be an operator satisfying hypotheses a)-c) of Theorem 7, and let  $R(\lambda, x, y, x', y')$  be the Schwarz kernel of  $(A + \lambda)^{-1}$ . The spectrum of  $A$  is  $[0, \infty]$ , so  $R(\lambda, x, y, x', y')$  is holomorphic outside the non-positive real axis. For any  $a > 0$ , let  $\Gamma_a$  be the path in  $\mathbb{C}$  consisting of two half-rays along  $\theta = -\varphi$  and  $\theta = \varphi$ , connected by the portion



FIGURE 4. Contour of integration  $\Gamma_a$ .

of the circle of radius  $a$  from  $\theta = -\varphi$  to  $\theta = \varphi$ , and traversed counterclockwise. Moreover, let  $\Gamma_{a,1}$  be the portion of  $\Gamma_a$  along the circle of radius  $a$ , and let  $\Gamma_{a,2}$  be the remainder; that is, the two half-rays. These contours are illustrated in Figure 4.

Let  $F(w, x, y, x', y')$  be the heat kernel at time  $t = w^{-2}$ . Then by the functional calculus, we have

$$F(w, x, y, x', y') = F(w, x, y, x', y') = \int_{\Gamma} e^{\lambda/w^2} R(\lambda, x, y, x', y') d\lambda. \quad (7)$$

We let  $a = w^2$  and  $\Gamma = \Gamma_{w^2}$ , and then consider the integral (7) over  $\Gamma_{w^2,1}$  and  $\Gamma_{w^2,2}$  separately.

On  $\Gamma_{w^2,1}$ ,  $\lambda = w^2 e^{i\theta}$ , so  $d\lambda = w^2 d\theta$ , and we have

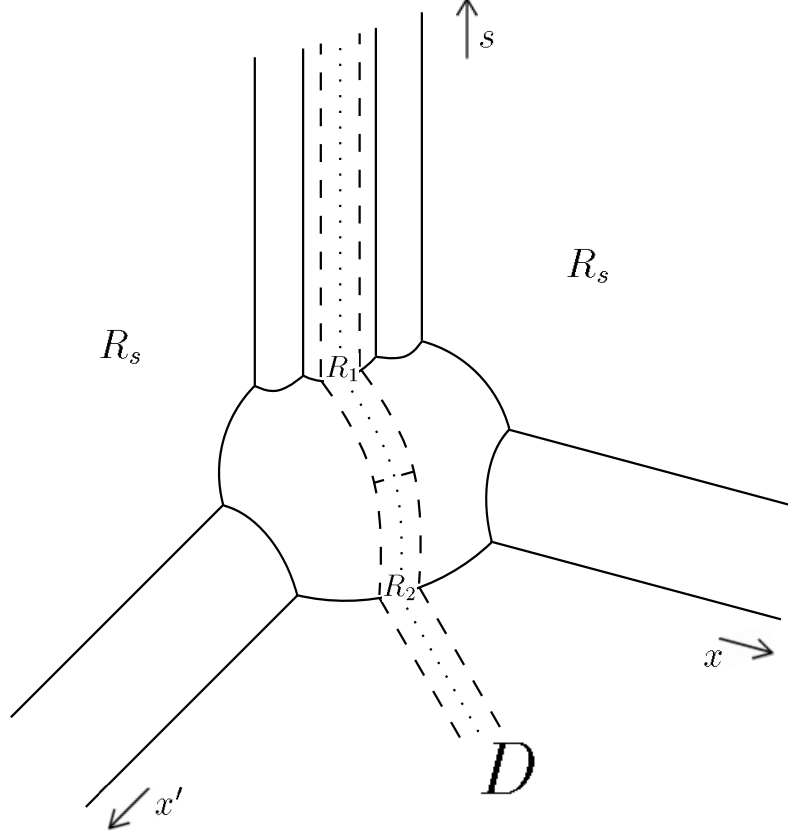
$$w^2 \int_{-\varphi}^{\varphi} e^{e^{i\theta}} R(\theta, w, x, y, x', y') d\theta. \quad (8)$$

By condition b), for each  $\theta$ , the integrand in (8) is phg conormal on  $M_{w,sc}^2$  with a conormal singularity at  $\Delta_{w,sc}$ , and the dependence of all coefficients on  $\theta \in [-\varphi, \varphi]$  is smooth. Therefore, the integral (8) is phg conormal on  $M_{w,sc}^2$ , with a possible conormal singularity at  $D_{w,sc}$ . The index sets of (8) on  $M_{w,sc}^2$  are those of  $w^2$  plus those of  $R(w, x, y, x', y')$ . The function  $w^2$  is smooth and has order 2 as a function at  $\text{zf}$ ,  $\text{bf}_0$ ,  $\text{rb}_0$ , and  $\text{lb}_0$  and order 0 everywhere else, so we add 2 to the index sets of the resolvent at those faces. This procedure gives precisely the index sets claimed in Theorem 7.

It remains to consider the integral over  $\Gamma_{w^2,2}$ .  $\Gamma_{w^2,2}$  consists of two half-rays; we consider only the half-ray corresponding to  $\theta = \varphi$ , as the other is analogous. Since  $\theta$  is fixed, we suppress  $\theta$  in the notation; the integral runs from  $r = w^2$  to  $r = \infty$ . After changing variables from  $r$  to  $s = \sqrt{r}$ , the  $\Gamma_{w^2,2}$  portion of (7) becomes:

$$\int_w^\infty 2s e^{(\cos \varphi)s^2/w^2} e^{i(\sin \varphi)s^2/w^2} R(s, x, y, x', y') ds. \quad (9)$$

First consider the behavior of the integrand as  $s \rightarrow \infty$ . By condition c),  $R(s, x, y, x', y')$  is polynomially bounded together with all derivatives as  $s \rightarrow \infty$ . Moreover, since  $\cos \varphi < 0$  and  $w$  is bounded above, the term  $\exp((\cos \varphi)s^2/w^2)$ , and hence the entire integrand of (9), decays exponentially together with all derivatives as  $s \rightarrow \infty$ .

FIGURE 5. Decomposition of  $R$ .

In order to analyze (9), we break  $R(s, x, y, x', y')$  into pieces. First we separate out the conormal singularity in a neighborhood of  $D_{s,sc}$  and analyze (9) using explicit local coordinates. Then we deal with the remainder, which is smooth on  $M_{s,sc}^2$ , by breaking it into two pieces and applying the pushforward theorem.

**2.4. The conormal singularity.** Using a partition of unity, we let  $R = R_s + R_c$ , where  $R_c$  is supported in a neighborhood of  $D_{s,sc}$  and  $R_s$  is supported away from  $D_{s,sc}$ , as in Figure 5.4. In a neighborhood of  $sc \cap bf_0$ , we use the coordinates

$$(X = s(\frac{1}{x} - \frac{1}{x'}), Y = s\frac{y - y'}{x}, y, \lambda = \frac{x}{s}, s).$$

In these coordinates,  $R$  has a conormal singularity at  $\{X = Y = 0\}$ . On the other hand, in a neighborhood of  $bf_0 \cap zf \cap D_{s,sc}$ , we use a slight modification of the coordinates in the previous subsection:

$$(\hat{X} = (1 - \frac{x}{x'}), \hat{Y} = y - y', y, \frac{s}{x}, x).$$

Since we use different coordinates in different regimes, write  $R_c = R_1 + R_2$  by using a smooth partition of unity near  $x/s = 1$ ;  $R_1$  is supported away from  $sc$  (say  $s/x < 2$ ) and  $R_2$  away from  $zf$  (say  $x/s < 2$ ). This decomposition is illustrated in Figure 5.

First look at  $R_1$ . Using the explicit symbolic form of a conormal singularity, we may write

$$R_1 \sim \int_{\mathbb{R}^n} e^{i(\hat{X}, \hat{Y}) \cdot (\xi_1, \xi_2)} \sum_{j=0}^{\infty} a_j\left(\frac{s}{x}, x, y, \frac{\xi}{|\xi|}\right) |\xi|^{2-j} d\xi. \quad (10)$$

This is an asymptotic sum, modulo smooth functions on  $M_{s,sc}^2$ ; we pick a particular representative which is supported in a small neighborhood of  $D_{s,sc}$ , and absorb the remainder into  $R_s$ . The coefficients  $a_j$  are phg conormal in  $x$  and  $s/x$  with index sets independent of  $j$ ; they are also smooth in  $y$  and  $\frac{\xi}{|\xi|}$ . We plug (10) into (9) and then interchange the convergent  $s$ -integral with the asymptotic sum and the oscillatory integral over  $\mathbb{R}^n$ . The result is

$$\int_{\mathbb{R}^n} e^{i(\hat{X}, \hat{Y}) \cdot (\xi_1, \xi_2)} \sum_{j=0}^{\infty} \left( \int_w^{\infty} e^{-(s/w)^2 e^{i\varphi}} a_j\left(\frac{s}{x}, x, y, \frac{\xi}{|\xi|}\right) ds \right) |\xi|^{2-j} d\xi. \quad (11)$$

By Melrose's pullback theorem [Me2], the pullback of each  $a_j$  to  $X_b^3(s, x, w) \times N_y \times S_{\xi/|\xi|}^{n-1}$  via projection is also phg conormal with index sets independent of  $j$ . As a result, the integrand in

$$\int_w^{\infty} e^{-(s/w)^2 e^{i\varphi}} a_j\left(\frac{s}{x}, x, y, \frac{\xi}{|\xi|}\right) ds \quad (12)$$

is phg conormal on  $X_b^3(s, w, x) \times N_y \times S_{\xi/|\xi|}^{n-1}$ , with a cutoff singularity at  $s/w = 1$  and with infinite-order decay as  $s \rightarrow \infty$ , together with all derivatives, independent of  $w < 1$ . Integration in  $s$  is a b-fibration from  $X_b^3(s, w, x)$  to  $X_b^2(w, x)$ , by Proposition 4.4 of [MS], and is hence also a b-fibration when we take the direct product with  $N_y \times S_{\xi/|\xi|}^{n-1}$ . Moreover, integration in  $s$  is transverse to the cutoff singularity at  $s/w = 1$ . Therefore, by the pushforward theorem with conormal singularities (from the Appendix of [EMM]), (12) is phg conormal on  $X_b^2(w, x) \times N_y \times S_{\xi/|\xi|}^{n-1}$ , with index sets independent of  $j$ .

Since  $w < s < 2x$  on the support of  $R_1$ , (12) has a phg conormal expansion in  $(w/x, x)$ . Therefore, the integral (11), in the coordinates  $(\hat{X}, \hat{Y}, y, w/x, x)$ , corresponding to the  $R_1$  piece of (9), is phg conormal in  $(w/x, x)$  with index sets independent of  $j$ , and smoothly dependent on  $y$ , with an interior conormal singularity at  $\hat{X} = \hat{Y} = 0$ . Thus (11) is phg conormal on  $M_{w,sc}^2$  with a conormal singularity at the diagonal.

We now consider  $R_2$ ; the analysis is similar. Write

$$R_2 \sim \int_{\mathbb{R}^n} e^{i(X, Y) \cdot (\xi_1, \xi_2)} \sum_{j=0}^{\infty} b_j\left(\frac{x}{s}, s, y, \frac{\xi}{|\xi|}\right) |\xi|^{2-j} d\xi, \quad (13)$$

where the  $b_j$  are phg conormal in  $\frac{x}{s}$  and  $s$  with index sets independent of  $j$ , and also smooth in  $y$  and  $\frac{\xi}{|\xi|}$ .

It is helpful to consider the regimes  $w > x/2$  and  $w < 2x$  separately. First assume that  $w > x/2$ , and let  $\bar{X} = w(\frac{1}{x} - \frac{1}{x'})$ ,  $\bar{Y} = w\frac{y-y'}{x}$ , and  $\bar{\lambda} = \frac{x}{w}$ . We expect a conormal singularity at  $\bar{X} = \bar{Y} = 0$  in this regime. Noting that  $(X, Y) = (s/w)(\bar{X}, \bar{Y})$ , we change variables in (13) and let  $\zeta = (s/w)\xi$ . The result is:

$$R_2 \sim \int_{\mathbb{R}^n} e^{i(\bar{X}, \bar{Y}) \cdot (\zeta_1, \zeta_2)} \sum_{j=0}^{\infty} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right) (s/w)^{j-2-n} |\zeta|^{2-j} d\zeta. \quad (14)$$

As before, plug (14) into (9) and interchange the sums and convergent integrals: the part of (9) coming from  $R_2$  is

$$\int_{\mathbb{R}^n} e^{i(\bar{X}, \bar{Y}) \cdot (\zeta_1, \zeta_2)} \sum_{j=0}^{\infty} \left( \int_w^{\infty} e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right) (s/w)^{j-2-n} ds \right) |\zeta|^{2-j} d\zeta. \quad (15)$$

Consider the coefficients

$$\int_w^{\infty} e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right) (s/w)^{j-2-n} ds. \quad (16)$$

If we can show that the coefficients (16) are phg conormal in  $(\frac{x}{w}, w)$  with respect to some index sets independent of  $j$ , with smooth dependence on  $y$  and  $\frac{\zeta}{|\zeta|}$ , then (15) is phg conormal on  $M_{w,sc}^2$  with a conormal singularity at the diagonal when  $w > x/2$ . To show this phg conormality, note that again the integrands in (16) are each phg conormal on  $X_b^3(s, x, w) \times N_y \times S_{\zeta/|\zeta|}^{n-1}$ . Moreover, the index sets are independent of  $j$ , as  $s/w > 1$ , and there is always infinite-order decay at  $s = \infty$ , independent of  $w$ . As before, we use the pushforward theorem to integrate in  $s$ , and we conclude that the coefficients (16) are phg conormal on  $X_b^2(x, w) \times N_y \times S_{\zeta/|\zeta|}^{n-1}$  with respect to index sets independent of  $j$ . Since  $w > x/2$ , this yields expansions in  $\frac{x}{w}$  and  $w$ , which is precisely what we need.

On the other hand, suppose that  $w < 2x$ . Then we expect a conormal singularity at  $\hat{X} = \hat{Y} = 0$ . Since  $(X, Y) = (s/x)(\hat{X}, \hat{Y})$ , we change variables in (13), letting  $\zeta' = (s/x)\zeta$ . Following the exact same procedure as in the  $w > x/2$  case, we see that the part of (9) coming from  $R_2$ , when  $w < 2x$ , is:

$$\int_{\mathbb{R}^n} e^{i(\hat{X}, \hat{Y}) \cdot (\zeta'_1, \zeta'_2)} \sum_{j=0}^{\infty} \left( \int_w^{\infty} e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta'}{|\zeta'|}\right) (s/x)^{j-2-n} ds |\zeta'|^{2-j} \right) d\zeta'. \quad (17)$$

The coefficients can be rewritten as

$$(w/x)^{j-2-n} \int_w^{\infty} e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right) (s/w)^{j-2-n} ds. \quad (18)$$

But these are just  $(w/x)^{j-2-n}$  times (16).  $(w/x)^{j-2-n}$  is phg conormal on  $X_b^2(w, x)$ , so (18) is also phg conormal on  $X_b^2(w, x)$  for each  $j$ . Since we are only considering  $w < 2x$ , the orders only improve as  $j$  increases. In particular, all the coefficients are phg conormal on  $X_b^2(w, x)$  with respect to subsets of the index set of the  $j = 0$  coefficient. This is again sufficient to prove that (17), and hence the part of (9) corresponding to  $R_c$ , is phg conormal on  $M_{w,sc}^2$  with a conormal singularity at the diagonal.

Technically, we need to compute the index sets of the coefficients of the conormal singularity at each boundary face of  $M_{w,sc}^2$ . This may be done directly via the pushforward theorem, but it is easier to apply the analysis we will develop in the next section. Note that  $a_j$  and  $b_j$  may be viewed as phg conormal functions on the diagonal  $D_{s,sc} \subset M_{s,sc}^2$ , with fixed index sets  $R_{sc}$ ,  $R_{bf_0}$ , and  $R_{zf}$  at the boundary hypersurfaces  $sc$ ,  $bf_0$ , and  $zf$ . Observe that they can be extended smoothly to functions defined in a neighborhood of  $D_{s,sc}$  which are themselves phg conormal on  $M_{s,sc}^2$  with the given index sets; call these extensions  $c_j$ . Then the coefficients of the conormal singularity of the integral (9) are the restrictions to the diagonal of

$$\int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} c_j(s, x, y, x', y') ds.$$

Applying Lemma 9 to each  $c_j$ , these coefficients are all phg conormal on  $M_{w,sc}^2$ , with index sets obtained by adding 2 at the faces  $bf_0$ ,  $rb_0$ ,  $lb_0$ , and  $zf$ . Therefore the restrictions to the diagonal are all phg conormal on  $D_{w,sc}$ , with leading orders matching those in Theorem 7, as expected. This completes the analysis of the conormal singularity.

**2.5. Finishing the proof.** It remains to consider the integral

$$\int_w^\infty 2se^{-(s/w)^2 e^{i\varphi}} R_s(s, x, y, x', y') ds, \quad (19)$$

where  $R_s(s, x, y, x', y')$  is phg conormal on  $M_{s,sc}^2$  and smooth across the diagonal. We claim:

**Lemma 9.** *Let  $T(s, x, y, x', y')$  be any function which is phg conormal on  $M_{s,sc}^2$ , smooth in the interior, decaying to infinite order at  $lb$ ,  $rb$ , and  $bf$ , and has at most polynomial growth in  $s$  as  $s \rightarrow \infty$  along with all derivatives (as in condition c) of Theorem 7). Then*

$$\int_w^\infty 2se^{-(s/w)^2 e^{i\varphi}} T(s, x, y, x', y') ds \quad (20)$$

*is phg conormal on  $M_{w,sc}^2$ . Moreover, if the index sets of  $T$  at the various boundary hypersurfaces are  $T_{sc}$ ,  $T_{bf_0}$ ,  $T_{zf}$ ,  $T_{rb_0}$ , and  $T_{lb_0}$ , then the index sets of (20) are:*

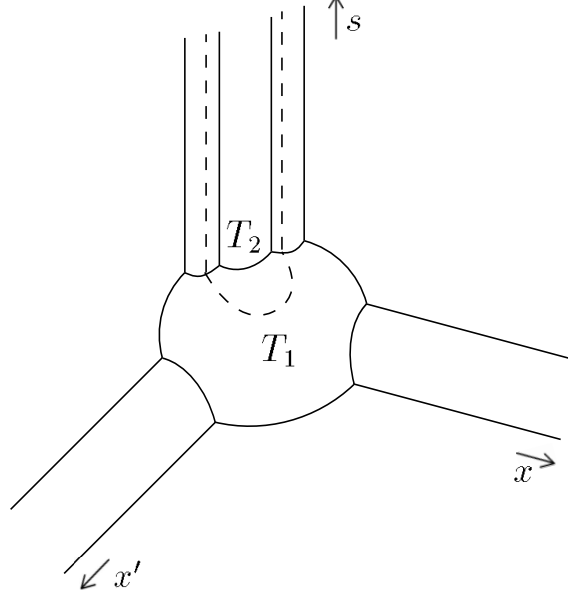
$$T_{sc}, T_{bf_0} + 2, T_{zf} + 2, T_{rb_0} + 2, T_{lb_0} + 2.$$

We defer the proof for the moment. Applying Lemma 9 to  $T = R_s$ , we conclude that (19) is phg conormal on  $M_{w,sc}^2$  with index sets precisely as in Theorem 7. Combining this with our analysis of  $R_c$ , we have now shown that  $F(w, x, y, x', y')$  is phg conormal on  $M_{w,sc}^2$  possibly with a conormal singularity at the spatial diagonal, and with leading orders as specified in Theorem 7. However,  $F(w, x, y, x', y')$  is a heat kernel, so it has no conormal singularity at the diagonal. This completes the proof of Theorem 7.

Finally, to prove Theorem 2, we apply Theorem 7. Condition a) is true since the Laplacian is essentially self-adjoint and non-negative. Condition b) follows from Theorems 5 and 6. Finally, condition c) is a well-known consequence of the semiclassical scattering calculus. The scattering calculus was first introduced by Melrose in [Me3] and the semiclassical version was developed by Vasy, Wunsch, and Zworski among others [VZ, WZ]. The exact statement we need, along with a summary of the semiclassical scattering calculus, may be found in section 10 of [HW]. Applying Theorem 7 gives us the polyhomogeneity we claim, and once we plug in the leading orders from [GH1] and the Appendix, we see that the heat kernel has leading orders of 0 at  $sc$  and  $n$  at each of  $bf_0$ ,  $rb_0$ , and  $lb_0$ .

Unfortunately, Theorem 7 does not by itself give us the claimed order- $n$  behavior at  $zf$ ; instead, we only see quadratic decay at  $zf$  when  $n \geq 3$  and decay of the form  $w^2 \log w$  when  $n = 2$ . However, by the estimate of Cheng-Li-Yau (Theorem 1), the heat kernel is uniformly bounded for large time by  $Ct^{-n/2} = Cw^n$ . Thus the leading order of the heat kernel at  $zf$  must actually be at least  $n$ , which completes the proof of Theorem 2. Notice that the order calculation in Theorem 7 is not sharp in this case. We expect that the lack of sharpness is due to some complex-analytic structure of the resolvent which is lost in our real-analytic approach.

**2.6. Proof of Lemma 9.** We now prove Lemma 9; the proof involves extensive use of Melrose's pullback and pushforward theorems. First, write  $T(s, x, y, x', y')$  as  $T_1 + T_2$ , where  $T_1$  is supported away from  $sc$  and  $T_2$  is supported in a neighborhood of  $sc$ . This partition is illustrated in Figure 6. Then decompose (20) into two integrals, corresponding to  $T_1$  and  $T_2$ .

FIGURE 6. Decomposition of  $T$ .

Consider the first integral:

$$\left( \int_0^\infty \chi(\{s \geq w\}) 2s^2 w x x' e^{-(s/w)^2 e^{i\varphi}} T_1(s, x, y, x', y') \frac{ds}{s} \right) \frac{dw}{w} \frac{dx}{x} \frac{dx'}{x'}. \quad (21)$$

Notice that  $T_1(s, x, y, x', y')$  is phg conormal on  $M_{s,sc}^2$  but supported away from  $sc$ , so it is in fact phg conormal on the blown-down space  $X_b^3(s, x, x') \times N_y \times N_{y'}$ . The rest of the terms in the integrand are phg conormal on  $X_b^2(s, w)$ , with a conormal singularity at  $s = w$ . By the pullback theorem, we conclude that the integrand is phg conormal on  $X_b^4(s, w, x, x') \times N_y \times N_{y'}$ , with a conormal singularity at  $s = w$ . It decays to infinite order as  $s \rightarrow \infty$  uniformly with all derivatives.

Now consider the map to  $X_b^3(w, x, x') \times N_y \times N_{y'}$  given by integration in  $s$ . This map is a b-fibration, with fibers that are transverse to the p-submanifold  $\{s = w\}$ , where there is a conormal singularity. So by the pushforward theorem, (21) is phg conormal on  $X_b^3(w, x, x') \times N_y \times N_{y'}$ . But  $X_b^3(w, x, x') \times N_y \times N_{y'}$  is a blow-down of  $M_{w,sc}^2(w, x, y, x', y')$ , so (21) is phg conormal on  $M_{w,sc}^2(w, x, y, x', y')$ , as desired.

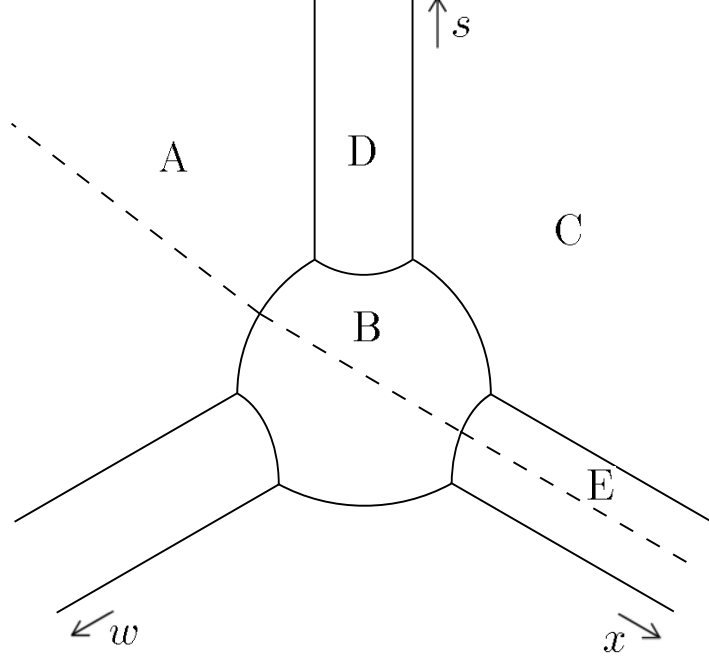
We now consider the integral

$$\left( \int_0^\infty \chi(\{s \geq w\}) 2s^2 w x x' e^{-(s/w)^2 e^{i\varphi}} T_2(s, x, y, x', y') \frac{ds}{s} \right) \frac{dw}{w} \frac{dx}{x} \frac{dx'}{x'}. \quad (22)$$

Let  $\bar{\sigma} = (\frac{x}{x'} - 1, y - y)$ ;  $\bar{\sigma}$  is an  $n$ -dimensional coordinate, and  $M_{s,sc}^2$  is created from  $X_b^3(s, x, x') \times N_y \times N_{y'}$  by blowing up  $\{\bar{\sigma} = \frac{x}{s} = 0\}$ . In particular,  $T_2(s, x, y, x', y')$ , having compact support in  $x/x'$ , is phg conormal on

$$[X_b^2(s, x) \times N_y; \{\bar{\sigma} = \frac{x}{s} = 0\}].$$

This space is the subset of  $M_{w,sc}^2$  with  $1/2 < x/x' < 2$ , so label its boundary hypersurfaces  $bf$ ,  $sc$ ,  $bf_0$ , and  $zf$ . In this labeling,  $T_2$  is supported away from  $zf$ , decays to infinite order at  $bf$ , and has leading orders  $t_{sc}$  at  $sc$  and  $t_{bf_0}$  at  $bf_0$ .

FIGURE 7. The space  $S$ .

We analyze the integrand in (22) as a function on the space

$$S = (X_b^3(s, w, x) \cap s \geq w) \times B(\bar{\sigma}) \times N_y.$$

Here  $B$  is the unit ball in  $\mathbb{R}^n$ . A diagram of  $S$  is given in Figure 7, with  $\bar{\sigma}$  and  $y$  suppressed; we label the boundary hypersurfaces A-E.

We now define an iterated blow-up of  $S$ . Let  $P_1$  be the p-submanifold of  $S$  given by  $A \cap \{\bar{\sigma} = 0\}$ . Blowing up  $P_1$  creates a new space  $S_1 = [S; P_1]$ ; call the front face of this blowup  $F$ . Now let  $P_2$  be the p-submanifold of  $S_1$  given by the closure of the lift of  $D^\circ \cap \{\bar{\sigma} = 0\}$ . Then let

$$S_2 = [S_1; P_2] = [[S; P_1]; P_2],$$

and let  $G$  be the new front face. The following two propositions allow us to analyze (22); their proofs are deferred for the moment.

**Proposition 10.** *The map*

$$\pi_w : S_2 \rightarrow [(X_b^2(s, x) \cap s \geq x) \times N_y; \{\bar{\sigma} = \frac{x}{s} = 0\}],$$

*given in the interior of  $S_2$  by projection off the variable  $w$  and extending continuously to the boundary, is a b-map.*

**Proposition 11.** *The map*

$$\pi_s : S_1 \rightarrow [X_b^2(w, x) \times B_1(\bar{\sigma} \times N_y; \{\bar{\sigma} = \frac{x}{w} = 0\}],$$

*given in the interior of  $S_1$  by projection off the variable  $s$  and extending continuously to the boundary, is a b-fibration. Moreover, if we let  $\rho_H$  be a bdf for each hypersurface  $H$ , we have*

$$(\pi_s)^*(\rho_{zf}) = \rho_C \rho_E, (\pi_s)^*(\rho_{bf_0}) = \rho_B \rho_D, (\pi_s)^*(\rho_{sc}) = \rho_F, (\pi_s)^*(\rho_{bf}) = \rho_A. \quad (23)$$

Since  $T_2$  is supported in  $\{s \geq x\}$ , Proposition 10 and the pullback theorem imply that the pullback of  $T_2$  is phg conormal on  $S_2$ . Moreover, the remainder of the integrand in (22) is phg conormal on  $(X_b^2(s, w) \cap \{s > w\})$ , so pulling back first to  $X_b^3(s, w, x) \cap \{s > w\}$ , and then to  $S_2$ , we see that it is phg conormal on  $S_2$  as well. Therefore, the entire integrand in (22) is phg conormal on  $S_2$ . However, since it is phg conormal on  $S_2 = [S_1; P_2]$  and vanishes to infinite order at the front face  $G$ , it is actually phg conormal on  $S_1$ . By the pushforward theorem from [EMM], since  $\pi_s$  is a b-fibration transverse to the conormal singularity at  $s = w$ , the pushforward (22) is phg conormal on the target space  $[X_b^2(w, x) \times B_1(\bar{\sigma} \times N_y; \{\bar{\sigma} = \frac{x}{w} = 0\})]$ . From Figure 9, we see that this space is a subset of  $M_{w,sc}^2$ ; we have therefore shown that (22) is phg conormal on  $M_{w,sc}^2$ . This completes the proof of the polyhomogeneity statement in Lemma 9, modulo the proofs of Propositions 10 and 11.

It remains to check the index sets claimed in Lemma 9. However, this calculation is a straightforward application of the pullback and pushforward theorems (explicit descriptions of the pullback and pushforward index sets may be found in [Gr]). A computation of the leading orders may be found in [S] and computing the index sets themselves is no harder.

**2.7. Propositions 10 and 11.** Finally, we prove Propositions 10 and 11. The proofs involve explicit local coordinates. For simplicity, redefine  $S$  to be  $X_b^3(s, w, x) \cap \{s \geq w\} \times B(\bar{\sigma})$ ; there is an extra  $N_y$ , but it is not involved in any of the blow-ups. Divide  $S$  into regions I-III, where  $s \geq w \geq x$  in region I,  $s \geq x \geq w$  in region II, and  $x \geq s \geq w$  in region III. This partition is illustrated in Figure 8. Projective coordinate systems in regions I-III are given by

$$(I) : (s, \frac{w}{s}, \frac{x}{w}, \bar{\sigma}), \quad (II) : (s, \frac{x}{s}, \frac{w}{x}, \bar{\sigma}), \quad (III) : (x, \frac{s}{x}, \frac{w}{s}, \bar{\sigma}). \quad (24)$$

Let  $\rho_A, \dots, \rho_E$  represent boundary defining functions for the boundary hypersurfaces. Each coordinate corresponds to a local bdf in the specified region, or to  $\sigma$ ; for example, in (II),  $s = \rho_B$ ,  $\frac{x}{s} = \rho_D$ , and  $\frac{w}{x} = \rho_C$ .

The new space  $S_1$  is created by blowing up  $A \cap \{\bar{\sigma} = 0\}$ . This blowup is localized to region I, so  $\rho_A = \frac{x}{w}$ . Call the new face  $F$ . To find coordinates on  $S_1$ , we split region I into two new regions: IV, where  $|\bar{\sigma}| < \frac{x}{w}$ , and V, where  $\frac{x}{w} < |\bar{\sigma}|$ . Then the coordinates (I) are replaced with two new coordinate systems:

$$(IV) : (s, \frac{w}{s}, \frac{x}{w}, \frac{\bar{\sigma}}{(x/w)}), \quad (V) : (s, \frac{w}{s}, \frac{(x/w)}{|\bar{\sigma}|}, \frac{\bar{\sigma}}{|\bar{\sigma}|}, |\bar{\sigma}|). \quad (25)$$

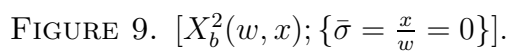
### *Proof of Proposition 11*

We now prove Proposition 11. We must show that projection off  $s$ , extended to the boundary in  $S_1$ , is a b-fibration onto  $[X_b^2(w, x); \{\bar{\sigma} = \frac{x}{w} = 0\}]$ . This space is illustrated in Figure 9; the faces are labeled  $sc$ ,  $bf$ ,  $bf_0$ , and  $zf$ , to illustrate that the space is a subset of  $M_{w,sc}^2$ . We split the space into three regions along the dotted lines in Figure 9; Region 1 is a neighborhood of  $zf$ , Region 2 is near the interior of  $sc$ , and Region 3 is the remainder (a neighborhood of  $bf$ ). Coordinates on  $[X_b^2(w, x); \{\bar{\sigma} = \frac{x}{w} = 0\}]$  are:

$$(1) : (x, \frac{w}{x}, \bar{\sigma}), \quad (2) : (w, \frac{x}{w}, \frac{\bar{\sigma}}{(x/w)}), \quad (3) : (w, \frac{(x/w)}{|\bar{\sigma}|}, \frac{\bar{\sigma}}{|\bar{\sigma}|}, \bar{\sigma}). \quad (26)$$

We first check that  $\pi_s$  is a b-map. This can be done using the local coordinate models. For example, region 1 is projected from regions II and III. Examining the map from region




$$(x, \frac{w}{x}, \bar{\sigma}) = (s \cdot \frac{x}{s}, \frac{w}{x}, \bar{\sigma}).$$

Hence every coordinate in region 1 is a product of (positive integer powers of) coordinates in region II. The same is true for region III, as

$$(x, \frac{w}{x}, \bar{\sigma}) = (x, \frac{w}{s} \cdot \frac{s}{x}, \bar{\sigma}).$$

Therefore  $\pi_s$  is a b-map onto region 1. Using this type of analysis for regions 2 and 3, it is easy to see that  $\pi_s$  is a b-map.

To check that  $\pi_s$  is a b-fibration, we look at the pullbacks of the boundary defining functions by using the local coordinate models. For example, in region 1,  $\rho_{zf} = \frac{w}{x}$ . The pullback to region II is  $\frac{w}{x} = \rho_C$ , and the pullback to region III is  $\frac{w}{s} \cdot \frac{s}{x} = \rho_C \rho_E$ . These are consistent, as region II is bounded away from E. So the pullback  $(\pi_s)^*(\rho_{zf}) = \rho_C \rho_E$ . We can perform this analysis on each face of the target space, and obtain:

$$(\pi_s)^*(\rho_{zf}) = \rho_C \rho_E, (\pi_s)^*(\rho_{bf_0}) = \rho_B \rho_D, (\pi_s)^*(\rho_{sc}) = \rho_F, (\pi_s)^*(\rho_{bf}) = \rho_A. \quad (27)$$

We conclude that

$$\pi_s(C) = \pi_s(E) = zf, \pi_s(B) = \pi_s(D) = bf_0, \pi_s(F) = sc, \pi_s(A) = zf.$$

In particular, no boundary hypersurface is mapped into a corner, so the first condition in the definition of b-fibration holds. As for the second (fibration) condition, the map from  $S$  to  $X_b^2(w, x) \times B(\bar{\sigma})$  given by integration in  $s$  is a b-fibration, by Propositions A.10 and A.12. Since the blow-up only involves faces A and F, we conclude that the fibration condition is satisfied on the interiors of  $sc$  and  $bf_0$ .

Finally, near A and F, we have coordinates (IV) and (V). The map from (IV) to (2) fixes the last two coordinates and maps  $(s, \frac{w}{s})$  to  $w = s \cdot \frac{w}{s}$ . Since the map from  $X_b^2(s, w)$  to  $\mathbb{R}_w^+$  is a b-fibration, so is the map from (IV) to (2). Analysis of the map from (V) to (3) is identical, so  $\pi_s$  is a b-fibration near A and F, and hence everywhere. This completes the proof of Proposition 11.

### *Proof of Proposition 10*

To prove Proposition 10, we need to make another blowup, of the submanifold  $P_2$  given by the closure in  $S_1$  of  $D^\circ \cap \{\bar{\sigma} = 0\}$ . As before, call the new face G. This new face intersects regions II and IV and requires new coordinates in each. In region II, we create G by blowing up  $\{\frac{x}{s} = 0, \bar{\sigma} = 0\}$ , and in region IV, we create G by blowing up  $\{\frac{w}{s} = 0, \frac{\bar{\sigma}}{(x/w)} = 0\}$ . Therefore, we split region IV into regions VI and VII, where  $\frac{\bar{\sigma}}{(x/w)} < \frac{w}{s}$  and  $\frac{\bar{\sigma}}{(x/w)} > \frac{w}{s}$  respectively; note that  $\frac{\bar{\sigma}/(x/w)}{(w/s)} = \frac{\bar{\sigma}}{(x/s)}$ . Also split region II into regions VIII and IX, where  $\bar{\sigma} < \frac{x}{s}$  and  $\frac{x}{s} < \bar{\sigma}$  respectively.

Then there are six different local coordinate systems on  $S_2$ : (III), (V), and:

$$(VI) : (s, \frac{w}{s}, \frac{x}{w}, \frac{\bar{\sigma}}{(x/s)}), \quad (VII) : (s, \frac{(x/s)}{|\bar{\sigma}|}, \frac{x}{w}, \frac{|\bar{\sigma}|}{(x/w)}, \frac{\bar{\sigma}}{|\bar{\sigma}|}),$$

$$(VIII) : (s, \frac{x}{s}, \frac{w}{x}, \frac{\bar{\sigma}}{(x/s)}), \quad (IX) : (s, \frac{(x/s)}{|\bar{\sigma}|}, \frac{w}{x}, \frac{\bar{\sigma}}{|\bar{\sigma}|}, |\bar{\sigma}|).$$

We want to show that  $\pi_w$  is a b-map onto  $[(X_b^2(s, x) \cap s \geq x); \{\bar{\sigma} = \frac{x}{s} = 0\}]$ . Since  $s > x$ , we can ignore region III. Following our analysis of the target space in Proposition 11, we break  $[(X_b^2(s, x) \cap s \geq x); \{\bar{\sigma} = \frac{x}{s} = 0\}]$  into two regions, region 2 (where  $|\bar{\sigma}| < \frac{x}{s}$ ) and

region 3 (everywhere else). Coordinates on these regions follow the previous analysis, with  $w$  replaced by  $s$ :

$$(2) : (s, \frac{x}{s}, \frac{\bar{\sigma}}{(x/s)}), \quad (3) : (s, \frac{(x/s)}{|\bar{\sigma}|}, \frac{\bar{\sigma}}{|\bar{\sigma}|}, \bar{\sigma}). \quad (28)$$

At this point, it is simply a matter of checking the local coordinate models to prove that  $\pi_w$  is a b-map. Note that regions VI and VIII map to region 2, whereas regions V, VII, and IX map to region 3. The details may be easily written down using the local coordinates as in the proof of Proposition 11.

It is worth mentioning that  $\pi_s$  is not a b-fibration from  $S_2$  to  $[X_b^2(w, x); \{\bar{\sigma} = \frac{x}{w} = 0\}]$ ;  $G$  gets mapped into the intersection of  $\text{bf}_0$  and  $\bar{\sigma} = 0$ .

### 3. RENORMALIZED HEAT TRACE AND ZETA FUNCTION

In this section, we define the renormalized heat trace, zeta function, and determinant on an asymptotically conic manifold  $M$ . The first step is to define the *renormalized trace*. This definition is inspired by Melrose's b-heat trace, which is a renormalized heat trace for manifolds with asymptotically cylindrical ends. Albin also defined renormalized heat traces in the asymptotically hyperbolic setting [Alb]; later, Albin, Aldana, and Rochon defined and investigated a renormalized determinant of the Laplacian on asymptotically hyperbolic surfaces [AAR].

**3.1. The renormalized heat trace.** Pick any cutoff function  $\chi_1(r)$  on  $\mathbb{R}_+$  which is supported on  $\{r \leq 2\}$  and equal to 1 on  $\{r \leq 1/2\}$ . Assume that either

- a)  $\chi_1(r)$  is a non-increasing smooth function of  $r$  (smooth cutoff), or
- b)  $\chi_1(r)$  is precisely the characteristic function of  $[0, 1]$  (sharp cutoff).

Then for any  $\delta < 1/2$ , let  $\chi_{1,\delta}$  be a function on  $M$ , equal to  $\chi_1(r\delta)$  for  $r \geq 1$  and equal to 1 inside  $\{r = 1\}$ . Consider the integral

$$\int_M \chi_{1,\delta}(z) H^M(t, z, z) dz. \quad (29)$$

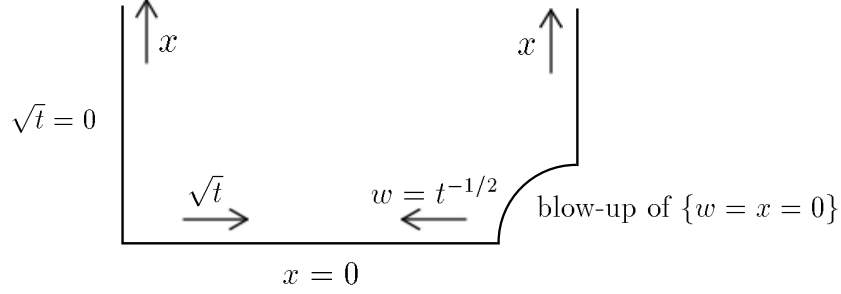
We will prove the following theorem:

**Theorem 12.** *Let  $\chi_1$  be either the smooth or the sharp cutoff. The integral (29) has a polyhomogeneous expansion in  $\delta$  for each fixed  $t$ . Moreover, the finite part at  $\delta = 0$ , which we denote  $P(t)$ , has polyhomogeneous expansions in  $t$  at  $t = 0$  and  $t^{-1}$  at  $t = \infty$ .*

This theorem allows us to define the renormalized heat trace on an asymptotically conic manifold. Roughly, this corresponds to integrating the heat kernel on the diagonal over regions where  $r \leq \delta^{-1}$ , and renormalizing by subtracting the divergent parts at  $\delta = 0$ . Renormalization in this fashion is often called Hadamard renormalization (for details, see [Alb, AAR]).

**Definition.** *Let  $\chi_1(r)$  be the sharp cutoff. The renormalized heat trace, denoted  ${}^R\text{Tr} H^M(t)$ , is the finite part at  $\delta = 0$  of (29).*

We now prove Theorem 12. The key ingredient is the following observation on the structure of the heat kernel on the diagonal, which is a consequence of the structure theorem we have proven for the heat kernel. The asymptotic structure of  $H^M(t, x, y, x, y)$  reflected in this proposition is illustrated in Figure 10.

FIGURE 10. Asymptotic structure of  $H^M(t, x, y, x, y)$ .

**Proposition 13.** *a) For  $t$  bounded above,  $H^M(t, x, y, x, y)$  is phg conormal in  $(\sqrt{t}, x)$ , with smooth dependence on  $y$ .*

*b) For  $t$  bounded below, let  $w = t^{-1/2}$ ; then  $H^M(t, x, y, x, y)$  is phg conormal as a function of  $w$  and  $x$  on  $X_b^2(w, x)$ , again with smooth dependence on  $y$ .*

*Proof.* Proposition 13 is an immediate consequence of restricting to the spatial diagonal  $D$  in Theorem 2; since  $D$  is a p-submanifold of the space on which  $H^M$  is polyhomogeneous, the restriction of  $H^M$  to  $D$  is also polyhomogeneous. Comparing  $D$  with Figure 10, we see that  $D$  is precisely the space described in Proposition 13.  $\square$

To prove Theorem 12, we analyze (29), which may be rewritten as

$$\int_N \int_0^1 \chi_1(\delta/x) H^M(t, x, y, x, y) x^{-n-1} dx dy + \int_{|z| \leq 1} H^M(t, z, z) dz. \quad (30)$$

First analyze the second term; the region  $\{|z| \leq 1\}$  is bounded away from spatial infinity. Therefore, by Proposition 13,  $H^M(t, z, z)$  has polyhomogeneous expansions in  $t$  at  $t = 0$  and  $t^{-1/2}$ , hence  $t^{-1}$ , at  $t = \infty$ , and these expansions are uniform in  $z$  with smooth coefficients. Integrating in  $z$  results in a function of  $t$  which is phg conormal at  $t = 0$  and  $t = \infty$ ; this function contributes only to the finite part  $P(t)$  at  $\delta = 0$  and satisfies the polyhomogeneity claimed in Theorem 12.

It remains to analyze the first term in (30). We consider the small- $t$  and large- $t$  regimes separately, analyzing the integrand

$$\chi_1(\delta/x) H^M(t, x, y, x, y) x^{-n-1} \quad (31)$$

in each regime as a function of  $(t, x, \delta)$ . In each case,  $\chi_1(\delta/x)$  is phg conormal on  $X_b^2(x, \delta)$ ; if  $\chi_1$  is the sharp cutoff, there is also a cutoff singularity, which is a type of conormal singularity, at  $\delta/x = 1$ .

For small  $t$ ,  $H^M(t, x, y, x, y)$  is phg conormal in  $(\sqrt{t}, x)$ , so (31) is phg conormal on  $\mathbb{R}_+(\sqrt{t}) \times X_b^2(x, \delta)$ , possibly with a conormal singularity at  $\delta/x = 1$ . The projection map  $\pi_x$  is a b-fibration from this space onto the first quadrant in  $(\sqrt{t}, \delta)$  and is transverse to  $\delta/x = 1$ ; moreover, the integral in  $x$  is well-defined, as the integrand is supported away from the  $x = 0, \delta > 0$  face. By the pushforward theorem from [EMM], the first term of (30) is phg conormal in  $(\sqrt{t}, \delta)$  for bounded  $t$ .

On the other hand, for large  $t$ ,  $H^M$  is phg conormal on  $X_b^2(w, x)$  and  $\chi_1$  is phg conormal on  $X_b^2(x, \delta)$ . Since the maps from  $X_b^3(w, x, \delta)$  to each of these spaces are b-maps (also b-fibrations), the integrand is phg conormal on  $X_b^3(w, x, \delta)$  by the pullback theorem; there may

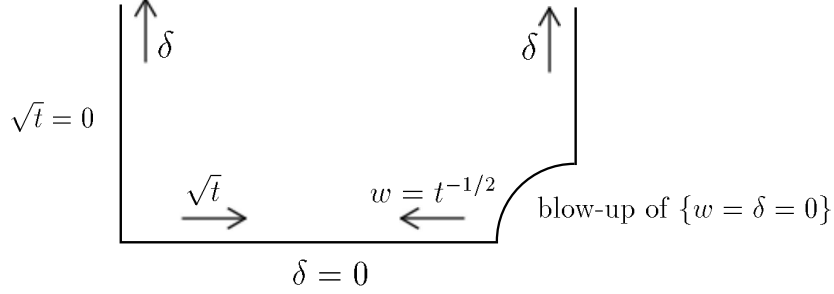


FIGURE 11. Asymptotic structure of (30).

again be a conormal singularity at  $\delta/x = 1$ . Integration in  $x$  is pushforward by a b-fibration onto  $X_b^2(w, \delta)$ . Again, the integrand is supported in  $\{x > \delta\}$ , and the fibration is transverse to  $\delta/x = 1$ , so we apply the pushforward theorem from [EMM] to conclude that the first term in (30) is phg conormal on  $X_b^2(w, \delta)$  for bounded  $w$ . Combining these results, we have shown that (30) is phg conormal on the space in Figure 11.

In particular, for any fixed  $t$ , (30) has a polyhomogeneous expansion as  $\delta \rightarrow 0$ . Moreover,  $P(t)$  is simply the coefficient of the  $t^0$  term at the  $(0 < t < \infty, \delta = 0)$  face. By the definition of phg conormality (also see the discussion surrounding Lemma A.4 in [Ma]),  $P(t)$  therefore has polyhomogeneous conormal expansions at  $t = 0$  and  $t = \infty$ . This completes the proof of Theorem 12.

Using Theorem 12, we now define the meromorphic continuation of the renormalized zeta function:

$${}^R\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty ({}^R\text{Tr} H^M)(t) t^{s-1} dt. \quad (32)$$

We break up the integral (32) at  $t = 1$ , and consider first the short-time piece:

$$\frac{1}{\Gamma(s)} \int_0^1 ({}^R\text{Tr} H^M(t)) t^{s-1} dt. \quad (33)$$

By the phg conormality of  ${}^R\text{Tr} H^M(t)$ , we can write for any  $N > 0$ :

$${}^R\text{Tr} H^M(t) = \sum_{i=0}^{k_N} a_i t^{z_i} (\log t)^{p_i} + \mathcal{O}(t^N).$$

Plug this expansion into (33). The  $\mathcal{O}(t^N)$  contribution is well-defined and meromorphic whenever  $\Re s > -N$ , and the continuations of the other terms are integrals of the form

$$\frac{a_i}{\Gamma(s)} \int_0^1 t^{z_i+s-1} (\log t)^{p_i} dt.$$

These integrals may be evaluated directly, and give explicit meromorphic functions of  $s$ , each with finitely many poles. Therefore, (33), though initially defined only when  $\Re s > -z_0$ , has a meromorphic extension to all of  $\mathbb{C}$ .

On the other hand, the long-time piece is

$$\frac{1}{\Gamma(s)} \int_1^\infty ({}^R\text{Tr} H^M(t)) t^{s-1} dt. \quad (34)$$

Writing  $u = 1/t$  and substituting, this becomes

$$\frac{1}{\Gamma(-(-s))} \int_0^1 ({}^R\text{Tr} H^M(1/u)) u^{(-s)-1} du.$$

We have a phg conormal expansion for  ${}^R\text{Tr} H^M(1/u)$  as  $u \rightarrow 0$ ; say the leading order term is of the form  $u^{z_\infty}(\log u)^p$ . Proceeding exactly as in the analysis of (33), we conclude that (34), though initially defined only when  $\Re(-s) > -z_\infty$ , has a meromorphic continuation to all of  $\mathbb{C}$ .

This allows us to define the renormalized zeta function and determinant on any asymptotically conic manifold  $M$ .

**Definition.** *The renormalized zeta function on  $M$ ,  ${}^R\zeta_M(s)$ , is given by the meromorphic continuation of (32).*

Depending on the orders, there may be no  $s$  in  $\mathbb{C}$  for which (32) is defined; however, once we split the integral at  $t = 1$ , both pieces are defined in half-planes and continue meromorphically to all of  $\mathbb{C}$ .

**Definition.** *The renormalized determinant of the Laplacian on  $M$  is  $e^{-{}^R\zeta'_M(0)}$ , where  ${}^R\zeta'_M(0)$  is the coefficient of  $s$  in the Laurent series for  ${}^R\zeta_M(s)$  at  $s = 0$ .*

**3.2. Manifolds conic near infinity.** We now specialize to the case of manifolds which are precisely conic outside a compact set. In particular, let  $Z$  be any asymptotically conic manifold without boundary which is isometric to a cone outside a compact set. Without loss of generality, assume that  $Z$  is isometric to a cone when  $r \geq 1/2$ . We examine the asymptotic expansion of  $\int_Z \chi_{1,\delta}(z) H^Z(t, z, z) dz$  as  $\delta \rightarrow 0$ ; the finite part is precisely the renormalized heat trace. However, for applications, such as the study of conic degeneration in [S2], we are also interested in identifying the divergent terms in the expansion. The fact that  $Z$  is conic near infinity allows us to identify those terms:

**Theorem 14.** *Let  $Z$  be conic near infinity as above, and let  $\chi_1$  be the sharp cutoff. Then  $\int_Z \chi_{1,\delta}(z) H^Z(t, z, z) dz$  has the following asymptotic expansion as  $\delta \rightarrow 0$ :*

$$\int_Z \chi_{1,\delta}(z) H^Z(t, z, z) dz = \sum_{k=0}^{n-1} f_k(t) \delta^{k-n} + f_{\log}(t) \log \delta + {}^R\text{Tr} H^Z(t) + R(\delta, t). \quad (35)$$

Here  $R(\delta, t)$  goes to zero as  $\delta$  goes to zero for each fixed  $t$ . Moreover, if we let  $u_k(1, y)$  be the coefficient of  $t^{(k-n)/2}$  in the short-time heat expansion on  $C_N$  at the point  $(1, y)$ , then  $f_k(t) = (\int_N u_k(1, y) dy) \frac{t^{(k-n)/2}}{k-n}$  and  $f_{\log}(t) = -\int_N u_n(1, y) dy$ .

We now prove Theorem 14. Note first that  $\int_Z \chi_{1,\delta}(z) H^Z(t, z, z)$  does in fact have a polyhomogeneous expansion in  $\delta$ , by Theorem 12, so it is just a matter of identifying the terms. The proof involves a comparison of the heat kernels on  $Z$  and on  $C_N$ ;  $Z$  and  $C_N$  are identical near infinity, which allows us to formulate and prove the following lemma:

**Lemma 15.** *Let  $C_N$  be the infinite cone over  $n$ . Then  $|H^{C_N}(t, z, z) - H^Z(t, z, z)|$ , defined whenever  $r \geq 1$ , decays to infinite order in  $|z|$  as  $|z|$  goes to infinity.*

*Proof.* Let  $\hat{Z} = Z \cap \{r \geq 1\}$ . It is a complete manifold with boundary at  $r = 1$ , and is a subset of both  $C_N$  and  $Z$ . On  $\hat{Z}$ ,  $H^{C_N}(t, z, z') - H^Z(t, z, z')$  is a solution of the heat equation for each  $z'$ , with initial data equal to zero and boundary data at  $r = 1$  given by

$H^{C_N}(t, 1, y, z') - H^Z(t, 1, y, z')$ . Fix any  $T > 0$ . We claim that for all  $t < T$ ,  $y \in N$ , and  $z'$  with  $|z'| > 2$ , there is a constant  $K$  so that the absolute value of the boundary data is less than  $K$ .

We show this for  $H^{C_N}(t, 1, y, z')$  and for  $H^Z(t, 1, y, z')$  separately. For  $C_N$ , by scaling and noting that  $|z'| < 2$ ,

$$H^{C_N}(t, 1, y, z') = |z'|^{-n} H^{C_N}(t/|z'|^2, 1/|z'|, y, z'/|z'|) < H^{C_N}(t/|z'|^2, 1/|z'|, y, 1, y').$$

For each fixed  $y'$ , the heat kernel with point source at  $(1, y')$  is continuous for  $r < 1/2$  (i.e. the tip of the cone), and hence is bounded for  $r < 1/2$  and for  $t/|z'|^2 < t < T$  by some universal constant  $K$ . Since  $y'$  varies only over a compact set, the proof is complete. As for  $H^Z(t, 1, y, z')$ , consider the region

$$W = \{(t, 1, y, x', y') \mid t < T, x' < \frac{1}{2}, y \in N, y' \in N\}$$

as a subset of  $(t, x, y, x', y')$  space. The kernel  $H^Z$  has infinite-order decay at each boundary hypersurface of the space in Figure 1 with which  $W$  has nontrivial intersection. We conclude that  $H^Z$  is bounded on  $W$ , so there is a constant  $K$  so that  $H^Z(t, 1, y, z') < K$  for all  $t < T$ , all  $y \in N$ , and all  $z'$  with  $|z'| > 2$ .

Since we have an upper bound for the boundary data, we can construct a supersolution and apply the parabolic maximum principle. Let  $g(t, r)$  be the solution of the heat equation on  $\hat{Z}$  with zero initial condition and boundary data at  $r = 1$  equal to  $K$  for all  $t$ . By the maximum principle, we see that, uniformly for  $|z'| > 2$  and  $t < T$ ,  $|H^{C_N}(t, z, z', t) - H^Z(t, z, z')| < g(t, |z'|)$ . We claim that  $g(t, r)$  decays to infinite order in  $r$ , uniformly in  $t$  for  $t < T$ . We can see this either directly from Bessel function expansions or by constructing a further supersolution  $\hat{g}(t, r)$  modeled on the heat kernel on  $\mathbb{R}^n$ . For completeness, we write down  $\hat{g}(t, r)$ :

$$\hat{g}(t, r) = \frac{K/\alpha}{(4\pi)^{n/2}} \sum_{k=-1}^{T_0} \frac{1}{(t-k)^{n/2}} e^{-\frac{r^2}{4(t-k)}} \chi_{\{t>k\}},$$

where  $\alpha = (8\pi)^{n/2} e^{1/4}$  and  $T_0$  is the greatest integer less than or equal to  $T$ . This supersolution has the uniform exponential decay property we want, so a final application of the parabolic maximum principle finishes the proof of the lemma.  $\square$

The following corollary is an immediate consequence:

**Corollary 16.** *For any fixed  $t$  and any  $\chi_{1,\delta}$  (either a sharp cutoff or a smooth cutoff),*

$$\left| \int_{C_N} \chi_{1,\delta}(z) H^{C_N}(t, z, z) dz - \int_Z \chi_{1,\delta}(z) H^Z(t, z, z) dz \right| \quad (36)$$

*converges as  $\delta \rightarrow 0$ .*

It now suffices to show that

$$\int_{C_N} \chi_{1,\delta}(z) H^{C_N}(t, z, z) dz$$

has a divergent asymptotic expansion of the form claimed in Theorem 14, as (36) converges as  $\delta \rightarrow 0$  and hence contributes only to the finite part of the expansion. (Recall that  $\chi$  is the sharp cutoff).

**Lemma 17.** *Fix  $t$ . The divergent terms in the expansion of  $\int_{|z| \leq 1/\delta} H^{C_N}(t, z, z)$  as  $\delta \rightarrow 0$  are given by:*

$$\sum_{k=0}^{n-1} \left( \int_N u_k(1, y) dy \right) \frac{t^{(k-n)/2}}{k-n} \delta^{k-n} - C \log \delta,$$

where  $C$  is equal to  $\int_N u_n(1, y) dy$ .

*Proof.* The integral is, modulo a term independent of  $\delta$ ,

$$\int_1^{1/\delta} \int_N H^{C_N}(t, r, y, r, y) r^{n-1} dy dr.$$

By the conformal homogeneity of  $C_N$ ,  $H^{C_N}(t, r, y, r, y) = r^{-n} H^{C_N}(t/r^2, 1, y, 1, y)$ . So the integral becomes:

$$\int_1^{1/\delta} \int_N H^{C_N}\left(\frac{t}{r^2}, 1, y, 1, y\right) r^{-1} dy dr.$$

Now let  $s = t/r^2$  and switch to an integral in  $s$ ; we get:

$$\frac{1}{2} \int_{\delta^2 t}^t \left( \int_N H^{C_N}(s, 1, y, 1, y) s^{-1} dy \right) ds. \quad (37)$$

From short-time heat asymptotics, we know that

$$\left( \int_N H^{C_N}(s, 1, y, 1, y) dy \right) = \sum_{k=0}^{n-1} \left( \int_N u_k(1, y) dy \right) s^{(k-n)/2} + \int_N u_n(1, y) dy + \mathcal{O}(s^{1/2}). \quad (38)$$

Here  $u_k(1, y)$  are the heat coefficients on the cone  $C_N$  at the point  $(1, y)$ . We plug (38) into (37) and get

$$\sum_{k=0}^{n-1} \left( \int_N u_k(1, y) dy \right) \frac{t^{(k-n)/2}}{k-n} \delta^{k-n} - \left( \int_N u_n(1, y) dy \right) \log \delta + g(\delta, t),$$

where  $g(\delta, t)$  is finite as  $\delta \rightarrow 0$ . This is what we wanted to prove.  $\square$

Combining this lemma with the preceding corollary and the definition of the renormalized heat trace completes the proof of Theorem 14.

Finally, it is also useful to investigate the analogous divergent expansion when a smooth cutoff, rather than a sharp cutoff, is used.

**Lemma 18.** *Let  $\chi_1(r)$  be as in a): smooth and non-increasing, supported in  $r \leq 2$  and 1 when  $r \leq 1/2$ . Then:*

$$\begin{aligned} \int_Z \chi_{1,\delta}(z) H^Z(t, z, z) dz &= \sum_{k=0}^{n-1} l_k f_k(t) \delta^{k-n} + f_{\log}(t) \log \delta \\ &+ ({}^R\text{Tr} H^Z(t) + l_{\log} f_{\log}(t)) + \tilde{R}(\delta, t), \end{aligned} \quad (39)$$

where  $l_k = -\int_{1/2}^2 \chi_1'(r) r^{k-n} dr$ ,  $l_{\log} = -\int_{1/2}^2 \chi_1'(r) \log r dr$ , and  $\tilde{R}(\delta, t)$  goes to zero as  $\delta$  goes to zero for every fixed  $t$ .



*Proof.* Let  $\xi(r)$  be any function which is equal to a constant  $a$  for  $r \leq 1/2$  and supported in  $\{r \leq 2\}$ . For any  $\delta < 1/2$ , we may define a function  $\xi_\delta(z)$  on  $Z$  by letting  $\xi_\delta(z)$  be equal to  $\xi\delta r$  for  $r = |z| \geq 1$  and  $a$  for  $\{r \leq 1\}$ . Then consider the integral

$$\int_Z \xi_\delta(z) H^Z(t, z, z) dz, \quad (40)$$

and examine its behavior as  $\delta \rightarrow 0$ .

When  $\xi(r)$  is the characteristic function of  $[0, 1]$ , we have the expansion (35). By replacing  $\delta$  with  $\delta/b$  for any  $b \in [1/2, 2]$ , we can compute the  $\delta \rightarrow 0$  expansion of (40) for  $\xi(r)$  equal to the characteristic function of  $[0, b]$ . By linearity, we see that the expansion of (40) for  $\xi(r) = (\Delta h)\chi_{[a,b]}$  is:

$$\begin{aligned} & \sum_{k=0}^{n-1} (\Delta h) f_k(t) (b^{k-n} - a^{k-n}) \delta^{k-n} + (\Delta h) f_{\log}(t) (\log b - \log a) \\ & + (\Delta h) (R(\delta/b, t) - R(\delta/a, t)). \end{aligned}$$

Now let  $\xi(r) = \chi_1(r) - \chi\{r \leq 1\}$ ; this is the difference between the sharp and smooth cutoffs. Since the expansions of (40) are linear in  $\xi$ , we can approximate by step functions and then integrate by summing over thin horizontal rectangles. Assume for simplicity that  $\chi_1(r) = 1$  for all  $r \leq 1$  (in the general case, there are some negative signs, but we get the same answer). The thickness of the rectangle at height  $h$  is  $\Delta h$ . The length of the rectangle is  $\chi_1^{-1}(h) - 1$ . Putting all of this together, the expansion of (40) with respect to  $\xi(r)$  is:

$$\sum_{k=0}^{n-1} f_k(t) \left( \int_0^1 ((\chi_1^{-1}(h))^{k-n} - 1) dh \right) \delta^{k-n} + f_{\log}(t) \left( \int_0^1 (\log \chi_1^{-1}(h)) dh \right) + \tilde{R}(\delta, t),$$

where  $\tilde{R}(\delta, t)$  is the contribution from the remainder terms.

Finally, perform the change of variables  $u = \chi_1^{-1}(h)$ , then add the expansion for  $\chi\{r \leq 1\}$ ; we obtain precisely the expansion claimed in the statement of the lemma. This finishes the proof, as long as we can control the remainder term  $\tilde{R}(\delta, t)$ . Indeed, for each fixed  $t$ , we claim that  $\tilde{R}(\delta, t)$  goes to zero as  $\delta$  goes to zero; define a new function  $S(\delta, t)$  by letting

$$S(\delta, t) = \sup_{1/2 \leq \gamma \leq 2} |R(\delta/\gamma, t)|.$$

When  $\xi(r) = (\Delta h)[a, b]$ , the remainder is bounded in absolute value by  $(\Delta h)S(\delta, t)$ . So the integral from  $h = 0$  to  $1$  is bounded by  $S(\delta, t)$ , which goes to zero as  $\delta \rightarrow 0$ ; this shows boundedness of the remainder term and finishes the proof of the lemma.  $\square$

It is worth examining the dependence on the zeta function on the choice of cutoff  $\chi_1$ ; we used a sharp cutoff to define it, but we could use a smooth cutoff instead. In this case, the finite part of the divergent  $\delta$ -expansion changes from  ${}^R\text{Tr}H^Z(t)$  to  ${}^R\text{Tr}H^Z(t) + l_{\log} f_{\log}(t)$ . But  $f_{\log}$  and  $l_{\log}$  are constants. So the renormalized heat trace only depends on the choice of cutoff function by the addition of a constant, independent of  $t$ . However, it can be easily shown by breaking up the integral at  $t = 1$  that the meromorphic continuation of  $\int_0^\infty C t^k t^{s-1} dt$  is identically zero for any constant  $C$ . We have shown:

**Proposition 19.** *Let  $Z$  be conic near infinity. The renormalized zeta function and determinant of the Laplacian on  $Z$  are independent of the choice of cutoff function  $\chi_{1,\delta}$ .*

We have now shown the existence of a renormalized zeta function and determinant of the Laplacian on any asymptotically conic manifold  $M$ ; moreover, when  $M$  is conic near infinity, we have computed the divergent terms in the expansion which leads to those renormalizations.

#### 4. THE LOW-ENERGY RESOLVENT IN TWO DIMENSIONS

In this section, we extend the techniques used by Guillarmou and Hassell in [GH1] to prove Theorem 6. In particular, we construct the low-energy resolvent on an asymptotically conic surface. The resolvent is

$$R(\theta, k, x, y, x', y') = (\Delta_M + e^{i\theta} k^2)^{-1}(x, y, x', y').$$

For simplicity, we set  $\theta = 0$ , so that  $R$  is a function of  $(k, x, y, x', y')$ . At the end of the section, we return to discuss allowing arbitrary  $\theta \in [-\varphi, \varphi]$  and showing smoothness in  $\theta$ ; however, this is not difficult.

**4.1. Strategy.** Our goal is to construct the Schwarz kernel of the resolvent,  $R(k, x, y, x', y')$ , as a distribution on  $M_{k,sc}^2$ . To do this, as in [GH1], we will first construct a parametrix  $G(k)$  so that  $(\Delta_M + k^2)G(k) = Id + E(k)$ , where  $E(k)$  is an error term.  $G(k)$  will be a family (in  $k$ ) of pseudodifferential operators on  $M$  whose Schwarz kernel is polyhomogeneous conormal on  $M_{w,sc}^2$  with an interior conormal singularity at the spatial diagonal. By examining the leading order behavior of the equation  $(\Delta_M + k^2)G(k) = Id$  at each boundary hypersurface of  $M_{w,sc}^2$ , we obtain a model problem at each hypersurface. The leading order of the parametrix  $G(k)$  at each hypersurface should solve the model problem. We first choose solutions of the model problem at each hypersurface, and then check that they are consistent; that is, that they may be glued together to obtain a parametrix  $G(k)$ . Finally, we analyze the error  $E(k)$  and show that it can be removed via a Neumann series argument.

In order to define the appropriate space of pseudodifferential operators, we use certain density conventions, all the same as in [GH1]. We consider  $P = \Delta_M$  as an operator on *scattering half-densities* by writing

$$P(f(x, y)|x^{-n-1}dxdy|^{1/2}) = (\Delta_M f)(x, y)|x^{-n-1}dxdy|^{1/2}.$$

As in [MSB] and [GH1], we expect a transition between scattering behavior for  $k > 0$  and b-behavior at  $k = 0$ , which leads us to define the conformally related  $b$ -metric  $g_b = x^2 g$ . The space  $(M, g_b)$  is asymptotically cylindrical. We then define  $P_b = x^{-n/2-1} P x^{n/2-1}$  with respect to scattering half-densities. However, we want to consider  $P_b$  acting with respect to  $b$ -half densities  $|x^{-1}dxdy|^{1/2}$ . After this shift, the relationship between  $P_b$  acting on  $b$ -half densities and  $P$  acting on scattering half-densities is  $P = x P_b x$ .

Let  $\tilde{\Omega}_b^{1/2}$  be the bundle of half-densities on  $M_{k,sc}^2$  which is spanned by sections of the form

$$|f(k, x, y, x', y') \frac{dg_b dg'_b dk}{k}|^{1/2}.$$

Let  $\nu$  be a smooth nonvanishing section of this bundle. Since it involves the  $b$ -metric  $g_b$ ,  $\tilde{\Omega}_b^{1/2}$  is not the natural bundle near  $sc$ . In particular, the kernel of the identity operator on  $M$  has leading order  $-n/2$  at  $sc$  with respect to  $\tilde{\Omega}_b^{1/2}$  [GH1]. We can now define spaces of pseudodifferential operators, precisely as in [GH1]:

**Definition.** Let  $\rho_{sc}$  be a boundary defining function for  $sc$ . The space  $\Psi_k^{m,\mathcal{E}}(M; \tilde{\Omega}_b^{1/2})$  is the space of half-density kernels  $K = K_1 + K_2$  on  $M_{k,sc}^2$  satisfying:

- 1)  $\rho_{sc}^{n/2} K_1$  is supported near  $\Delta_{k,sc}$ , and has an interior conormal singularity of order  $m$  at  $\Delta_{k,sc}$ , with coefficients whose behavior at the boundary is specified by  $\mathcal{E}$ ;
- 2)  $\rho_{sc}^{n/2} K_2$  is polyhomogeneous conormal on  $M_{k,sc}^2$  with index family  $\mathcal{E}$ , and moreover decays to infinite order at  $bf$ ,  $lb$ , and  $rb$ .

The factor of  $\rho_{sc}^{n/2}$  corrects for the use of b-half densities near  $sc$ . Using this definition, we can compute as in [GH1] that

$$(P + k^2) \in \Psi_k^{2,\mathcal{E}}(M; \tilde{\Omega}_b^{1/2}),$$

with index sets 0 at  $sc$ , 2 at  $bf_0$ , 0 at  $zf$ , 2 at  $lb_0$ , and 2 at  $rb_0$ .

As proven in [GH1], these spaces satisfy a composition rule:

**Proposition 20.** Suppose that  $A \in \Psi_k^{m,\mathcal{E}}$  and  $B \in \Psi_k^{m',\mathcal{F}}$ . Then  $A \circ B$  is well defined and an element of  $\Psi_k^{m+m',\mathcal{G}}$ , where

$$\begin{aligned} \mathcal{G}_{sc} &= \mathcal{E}_{sc} + \mathcal{F}_{sc}, \mathcal{G}_{zf} = (\mathcal{E}_{zf} + \mathcal{F}_{zf}) \cup (\mathcal{E}_{rb_0} + \mathcal{F}_{lb_0}), \mathcal{G}_{bf_0} = (\mathcal{E}_{bf_0} + \mathcal{F}_{bf_0}) \cup (\mathcal{E}_{lb_0} + \mathcal{F}_{rb_0}), \\ \mathcal{G}_{lb_0} &= (\mathcal{E}_{bf_0} + \mathcal{F}_{lb_0}) \cup (\mathcal{E}_{lb_0} + \mathcal{F}_{zf}), \mathcal{G}_{rb_0} = (\mathcal{E}_{rb_0} + \mathcal{F}_{bf_0}) \cup (\mathcal{E}_{zf} + \mathcal{F}_{rb_0}). \end{aligned}$$

We therefore expect our parametrix  $G$  to be in  $\Psi_k^{-2,\mathcal{G}}$  for some index family  $\mathcal{G}$ . To gain more information, we need to start analyzing the model problems.

**4.2. The two-dimensional problem.** We begin our analysis of the model problems at the face  $zf$ . In order to identify the leading-order part of the equation at a boundary hypersurface, we need to pick a coordinate to use as a boundary defining function in the interior of that face. For all the faces in the lift of  $\{k = 0\}$ , we use  $k$ , which is the easiest choice, since it commutes with  $P$ . Since  $P + k^2 = xP_b x + k^2$ , the leading order part of the operator at  $zf$ , which we call the *normal operator*, is  $xP_b x$ , and the model problem is  $(xP_b x)G_{zf}^0 = Id$ . We therefore expect that  $G_{zf}^0$  will be  $(xx')^{-1}$  times some right inverse for  $P_b$ .

In order to invert  $P_b$ , we use the b-calculus of Melrose [Me], identifying  $zf$  with the b-double space  $X_b^2(x, x') \times N_y \times N_{y'}$ . The corner  $zf \cap bf_0$  corresponds to the front face  $ff$  in the b-double space. An easy calculation, following [GH1], shows that

$$P_b = -(x\partial_x)^2 + (n/2 - 1)^2 + \Delta_N + W,$$

where  $W$  is a lower-order term; that is,  $W$  vanishes as a b-differential operator at  $x = 0$ . In fact,  $P_b$  is an elliptic b-differential operator, and hence may be inverted by following the procedure of Melrose, which is described in [Me] and [Ma].

The first step in this procedure is to consider the indicial operator, which is the leading order part of  $P_b$  at the front face  $ff$ . Using the coordinates  $(s, x, y, y')$ , this is

$$I_{ff}(P_b) = -(s\partial_s)^2 + (n/2 - 1)^2 + \Delta_N.$$

With this terminology, the key theorem is as follows:

**Theorem 21.** [Me]  $P_b$  is Fredholm as an operator between  $x^\delta H_b^2$  and  $x^\delta L_b^2$  if and only if  $\delta$  is not an indicial root of  $I_{ff}(P_b)$ . (See [Me] or [Ma] for definitions of  $x^\delta H_b^2$  and  $x^\delta L_b^2$ ).

In our setting, as in [GH1], the indicial roots are precisely

$$\pm\nu_i = \pm\sqrt{(n/2 - 1)^2 + \lambda_i}; \lambda_i \in \sigma(\Delta_N).$$

When  $n > 2$ , 0 is not an indicial root, and Guillarmou and Hassell show that  $P_b$  is not only Fredholm but invertible for  $\delta = 0$ , and then set  $G_{zf}^0$  to be  $(xx')^{-1}$  times that inverse. However, in our case,  $n = 2$ , so  $\nu_i = \sqrt{\lambda_i}$ , and 0 is an indicial root. So  $P_b$  is not even Fredholm from  $H_b^2$  to  $L_b^2$ . This is precisely why the  $n = 2$  case is not considered in [GH1].

**4.3. An example: Euclidean space.** In order to gain some intuition for the behavior of the resolvent near zf in the  $n = 2$  setting, we examine the simplest case, which is  $M = \mathbb{R}^2$ . The resolvent on  $\mathbb{R}^2$ , acting on scattering half-densities  $|dz| = |x^{-3}dxdy|$ , is

$$-\frac{1}{2\pi}H_0(k|z - z'|) = H_0(k|\frac{y}{x} - \frac{y'}{x'}|),$$

where  $H_0$  is the Hankel function of order zero. From the asymptotics of the Hankel function, we know that  $H_0(r)$  decays exponentially as  $r \rightarrow \infty$ , and for small  $r$ :

$$H_0(r) \sim -\log r + \log 2 - \gamma + \mathcal{O}(r).$$

Using these asymptotics, one can show that the resolvent on  $\mathbb{R}^2$  is phg conormal on  $M_{k,sc}^2$ . We are most interested in the leading order behavior near zf. In a neighborhood of zf, we have  $k|z - z'| < 1$ , so this leading order behavior is controlled by the small- $r$  asymptotics

$$H_0(k|z - z'|) \sim -\log(k|z - z'|) + \log 2 - \gamma = -\log k - \log|z - z'| + \log 2 - \gamma.$$

Some observations on these asymptotics:

- As we approach zf, the resolvent increases logarithmically. This is a major difference from the  $n \geq 3$  case studied in [GH1], in which the resolvent is continuous down to zf. On the other hand, the resolvent is continuous down to  $\text{bf}_0$ ,  $\text{lb}_0$ , and  $\text{rb}_0$ .

- The function  $-\frac{1}{2\pi}\log|z - z'|$  is the Green's function for the Laplacian on  $\mathbb{R}^2$ .

These observations suggest that the leading order behavior in the general situation may be of the form

$$G_{zf}^0 = -C\log k + F(x, y, x', y'),$$

where  $F(x, y, x', y')$  is a right inverse for the operator  $P$  with respect to scattering half-densities. Moreover, since  $-C\log k$  has logarithmic growth at  $\text{bf}_0$ ,  $\text{lb}_0$ , and  $\text{rb}_0$  but the resolvent on  $\mathbb{R}^2$  does not, we expect that  $F(x, y, x', y')$  will have logarithmic growth at those faces, with the right coefficient to cancel the logarithmic growth coming from  $-C\log k$ .

**4.4. Construction of the initial parametrix.** We now construct our parametrix  $G(k)$  by specifying its leading order behavior at each boundary hypersurface and then checking that the models are consistent.

**4.4.1. The diagonal, sc, and  $\text{bf}_0$ .** The resolvent has an interior conormal singularity at the diagonal  $\{x = x', y = y'\}$ . The symbol of  $P + k^2$  is  $|\xi|^2 + k^2$ , where  $\xi$  is the dual variable of  $|z - z'|$ . One can compute that  $|\xi|^2 + k^2$  is elliptic on  $M_{k,sc}^2$ , with leading orders 0 at sc, 2 at  $\text{bf}_0$ , and 0 at zf. As in [GH1], we let the symbol of  $G(k)$  be the inverse of  $|\xi|^2 + k^2$ , in the sense of operator composition. This determines the diagonal symbol of  $G(k)$  up to symbols of order  $-\infty$ , and hence determines  $G(k)$  up to operators with smooth Schwarz kernels in the interior of  $M_{w,sc}^2$ .

At sc, the analysis is identical to that in [GH1], so we omit some of the details. The key point is that sc can be described as a fiber bundle with  $\mathbb{R}^n$  fibers, parametrized by  $y' \in N$

and  $k$ . The normal operator of  $P + k^2$  is  $\Delta_{\mathbb{R}^n} + k^2$ , which has a well-defined inverse for  $k > 0$ . In each fiber, we let

$$G_{sc}^0 = (\Delta_{\mathbb{R}^2} + k^2)^{-1}.$$

At  $\text{bf}_0$ , we again follow [GH1] exactly. We use the coordinates  $(\kappa = k/x, \kappa' = k/x', y, y')$ , with  $k$  a bdf for  $\text{bf}_0$ . Note that these are only good coordinates on the interior of  $\text{bf}_0$  - for example, they become degenerate near  $\text{zf}$ . We then view the interior of  $\text{bf}_0$  as  $\mathbb{R}_+(\kappa) \times \mathbb{R}_+(\kappa') \times N_y \times N_{y'}$ . As in [GH1], the normal operator at  $\text{bf}_0$  is

$$I_{\text{bf}_0}(k^{-2}(P + k^2)) = \kappa^{-1}(-(\kappa\partial_\kappa)^2 + \Delta_N + \kappa^2)\kappa^{-1}.$$

Letting  $P_{\text{bf}_0} = -(\kappa\partial_\kappa)^2 + \Delta_N + \kappa^2$ , the model problem is  $(\kappa P_{\text{bf}_0} \kappa) G_{\text{bf}_0}^{-2} = \delta\kappa - \kappa'\delta - y'$ . To solve it, we separate variables and invert  $P_{\text{bf}_0}$ . For each eigenvalue  $\lambda_j$  of  $\Delta_N$ , write  $\nu_j = \sqrt{\lambda_j}$  (these are the indicial roots). Let  $E_{\nu_j} \subset L^2(N)$  be the corresponding eigenspace of  $\Delta_N$ , and let  $\Pi_{E_{\nu_j}}$  be projection in  $L^2(N)$  onto  $E_{\nu_j}$ . Then the inverse of  $P_{\text{bf}_0}$  is

$$Q_{\text{bf}_0} = \sum_{j=0}^{\infty} \Pi_{E_{\nu_j}} (I_{\nu_j}(\kappa) K_{\nu_j}(\kappa') \chi_{\{\kappa' > \kappa\}} + I_{\nu_j}(\kappa') K_{\nu_j}(\kappa) \chi_{\{\kappa' < \kappa\}}).$$

The only difference between our setting and [GH1] is that we have  $\nu_0 = 0$  as opposed to  $\nu_0 > 0$ . We then set

$$G_{\text{bf}_0}^{-2} = (\kappa\kappa') Q_{\text{bf}_0}.$$

We need to check consistency between  $G_{sc}^0$  and  $G_{\text{bf}_0}^{-2}$ ; that is, we need to show that they agree to leading order in a neighborhood of  $\text{sc} \cap \text{bf}_0$ . This proof is the same as in [GH1]; the model problems and formal expressions for  $G_{sc}^0$  and  $G_{\text{bf}_0}^{-2}$  are identical. We do have  $\nu_0 = 0$ , and  $K_0(r)$  has different small- $r$  asymptotics from  $K_{\nu_j}(r)$  for  $\nu_j > 0$ ; this will be reflected in the asymptotics of  $G_{\text{bf}_0}^{-2}$  near  $\text{zf}$ . However, since  $\kappa$  and  $\kappa'$  both approach infinity near  $\text{sc}$ , only the large- $r$  asymptotics are relevant for this consistency check, and  $K_0(r)$  decays exponentially just as  $K_{\nu_j}(r)$  does.

Technically, we also need to check consistency between the diagonal symbol and the models at  $\text{bf}_0$  and  $\text{sc}$ . However, this is also the same as in [GH1]; the models at  $\text{bf}_0$  and  $\text{sc}$  themselves satisfy elliptic pseudodifferential equations given by the leading order part of  $(P + k^2)G(k) = \text{Id}$  at those faces. As a result, their symbols at the diagonal are determined up to symbols of order  $-\infty$ , and agree up to order  $-\infty$  with the inverse of  $|\xi|^2 + k^2$ .

**4.4.2. The leading order term at  $\text{zf}$ .** At  $\text{zf}$ , the model problem with respect to b-half-densities is:

$$(xP_b x) G_{\text{zf}}^0 = \text{Id}.$$

Translating our observations in the  $M = \mathbb{R}^2$  case to b-half-densities, we expect

$$G_{\text{zf}}^0 = (xx')^{-1}(-C \log k + F(x, y, x', y')),$$

where  $F(x, y, x', y')$  is a right inverse for  $P_b$ . We need to pick the correct right inverse, and we expect that it will have logarithmic singularities at all faces. We also need it to be consistent with our choice of  $G_{\text{bf}_0}^{-2}$ . To check consistency, we need to show that  $k^0 G_{\text{zf}}^0$  and  $k^{-2} G_{\text{bf}_0}^{-2}$  agree to leading order at  $\text{ff} = \text{bf}_0 \cap \text{zf}$ , which is the same as checking if  $(xx') G_{\text{zf}}^0$  and  $Q_{\text{bf}_0} = (\kappa\kappa')^{-1} G_{\text{bf}_0}^{-2}$  agree to leading order there.

First examine the leading order part of  $Q_{bf_0}$  at zf. When  $x < x'$ , we use the coordinates  $(s, \kappa, x', y, y')$ , and we have

$$Q_{bf_0} = V^{-1}I_0(\kappa s)K_0(\kappa) + \sum_{j=1}^{\infty} \Pi_{E_j} I_{\nu_j}(\kappa s) K_{\nu_j}(\kappa). \quad (41)$$

The boundary defining function for zf is  $\kappa$ , so we need to examine the small- $\kappa$  asymptotics. For  $\nu > 0$  we know by standard asymptotics of Bessel functions in [Wa] that

$$I_0(r) \sim 1; K_0(r) \sim -\ln r + \ln 2 - \gamma; \\ I_{\nu}(r) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{r}{2}\right)^{\nu}; K_{\nu}(r) \sim \frac{\Gamma(\nu)}{2}\left(\frac{r}{2}\right)^{-\nu}.$$

Here  $\gamma$  is the Euler-Mascheroni constant. Plugging these asymptotics into (41) shows that the leading order term in  $\kappa$  is:

$$V^{-1}(-\log \kappa + \log 2 - \gamma) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{s^{\nu_j}}{2\nu_j}.$$

On the other hand, when  $x > x'$ , we use the coordinates  $(s', \kappa', x, y, y')$  and perform the same sort of calculations to obtain that the leading order term in  $\kappa$  is

$$V^{-1}(-\log \kappa' + \log 2 - \gamma) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{(s')^{\nu_j}}{2\nu_j}.$$

The  $x < x'$  and  $x > x'$  cases may be combined; we see that the leading order term of  $Q_{bf_0}$  at zf is

$$V^{-1}(-\log k + \log 2 - \gamma + \log x' + \chi_{s < 1} \log s) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{e^{-\nu_j |\log s|}}{2\nu_j}. \quad (42)$$

We need to construct  $G_{zf}^0$  so that  $(xx')G_{zf}^0$  has this leading order at  $bf_0$ .

To construct  $G_{zf}^0$ , we must first find the correct right inverse for  $P_b$ . Fix  $\delta$  with  $0 < \delta < \nu_1$ ; then  $P_b$  is Fredholm from  $x^{-\delta}H_b^2$  to  $x^{-\delta}L_b^2$  by Theorem 21. Following the usual b-calculus construction in [Me] and [Ma], we obtain a generalized inverse  $Q_b^{-\delta}$ . We claim:

**Lemma 22.**  *$P_b$  is surjective onto  $x^{-\delta}L_b^2$ .*

The lemma implies that  $Q_b^{-\delta}$  is an exact right inverse for  $P_b$ .

*Proof.* By taking adjoints, the lemma is equivalent to the statement that  $P_b$  is injective on  $x^{\delta}L_b^2$ . Suppose that  $u|dg_b|^{1/2}$  is in  $x^{\delta}L_b^2$  and satisfies  $P_b u = 0$ . By regularity of solutions to b-elliptic equations,  $u$  is phg conormal on  $M$  near  $x = 0$ ; since  $u \in x^{\delta}L_b^2$ , it decays to at least order  $\delta$  at  $x = 0$ . On the other hand, since  $P = xP_b x$  and  $|dg_b|^{1/2} = x|dg|^{1/2}$ ,  $u|dg|^{1/2}$  is in the kernel of  $P = \Delta_M$ , and hence  $\Delta_M u = 0$ . By the maximum principle,  $u = 0$ , which completes the proof of the lemma.  $\square$

The correct right inverse will be a slight modification of  $Q_b^{-\delta}$ . In order to check consistency, we need to understand the structure of  $Q_b^{-\delta}$  near the front face  $\text{ff} = \text{bf}_0 \cap \text{zf}$ . This structure is described in detail in [Me] and [Ma]. In particular, the leading order of  $Q_b^{-\delta}$  at ff is precisely the indicial operator  $I_{ff}(Q_b^{-\delta})$ , which satisfies the equation

$$-(s\partial_s)^2 + \Delta_N I_{ff}(Q_b) = \delta(s = 1, y = y'). \quad (43)$$

Moreover, from [Me] and [Ma],  $I_{ff}(Q_b^{-\delta})$  has polyhomogeneous expansions at  $s = 0$  and  $s = \infty$ , with leading order terms at worst  $s^{-\delta}$  at each end; that is, a small amount of growth is allowed at  $s = 0$ , and a small amount of decay is required at  $s = \infty$ .

We now separate variables and solve (43) directly. For each  $j \geq 1$ ,  $s^{\pm\nu_j}$  span the kernel of  $-(s\partial_s)^2 + \nu_j^2$ . Therefore, the solutions corresponding to  $E_{\nu_j}$  are combinations of  $s^{\nu_j}$  and  $s^{-\nu_j}$  away from  $s = 1$ . By the requirements at  $s = 0$  and  $s = \infty$ , our solution is a multiple of  $s^{-\nu_j}$  for  $s > 1$  and of  $s^{\nu_j}$  for  $s < 1$ . Using the matching conditions at  $s = 1$  arising from the delta function singularity, the solution on the eigenspace  $E_{\nu_j}$  is

$$\Pi_{E_{\nu_j}} \frac{e^{-\nu_j |\log s|}}{2\nu_j}.$$

We have to consider  $\nu_0 = 0$  separately; the kernel of  $-(s\partial_s)^2$  is spanned by 1 and  $\log s$ . Because we require decay at  $s = \infty$ , the solution for  $s > 1$  must be zero. Then the matching conditions at  $s = 1$  imply that the solution is  $\log s$  for  $s < 1$ . Since projection onto  $E_0$  is simply  $V^{-1}$ , where  $V = \text{Vol}(N)$ , the zero-eigenspace solution is  $V\chi_{\{s < 1\}} \log s$ . Therefore the leading order part of  $Q_b^{-\delta}$  at ff is

$$I_{ff}(Q_b^{-\delta}) = V^{-1}\chi_{\{s < 1\}} \log s + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{e^{-\nu_j |\log s|}}{2\nu_j}. \quad (44)$$

Now compare (44) with (42). We see immediately that if we let

$$G_{zf}^0 = (xx')^{-1}(-V^{-1} \log k + Q_b^{-\delta} + V^{-1} \log x' + V^{-1}(\log 2 - \gamma)),$$

then  $G_{zf}^0$  and  $G_{bf_0}^{-2}$  are consistent. In addition, since any function independent of  $x$  and  $y$  is in the kernel of  $P_b$ ,  $G_{zf}^0$  solves the model problem  $(xP_b x)G_{zf}^0 = Id$  at zf. Moreover, the diagonal symbol is consistent with  $G_{zf}^0$  for the same reason that it is consistent with  $G_{sc}^0$  and  $G_{bf_0}^{-2}$ .

**4.4.3. The model terms at  $rb_0$ .** Finally, we need to specify the leading-order behavior of the parametrix at  $rb_0$ ; in fact, we need to specify some lower-order terms as well. We use the coordinates  $(x, y, \kappa', y')$ ; the  $\kappa' = 0$  face is  $rb_0 \cap zf$  and the  $x = 0$  face is  $rb_0 \cap bf_0$ . There will be a term  $G_{rb_0}^{\nu_j - 1}$  for each  $\nu_j$  in  $[0, 1)$ . The model problem near this face, with  $k$  as a boundary defining function, is  $(xP_b x)u = 0$ , so we need  $P_b(xG_{rb_0}^{-\nu_j}) = 0$  for each  $\nu_j \in [0, 1)$ .

First we focus on the model of order  $-1$ . We let

$$G_{rb_0}^{-1} = V^{-1}x^{-1}\kappa'K_0(\kappa'),$$

and claim that this is consistent with  $G_{zf}^0$  and  $G_{bf_0}^{-2}$ .

To check consistency with  $G_{zf}^0$ , we need to show that the leading order of  $G_{zf}^0$  agrees with the leading order of  $k^{-1}G_{rb_0}^{-1}$  at  $zf \cap rb_0$ . Recall that at  $rb_0$ , which corresponds to  $s = \infty$ ,  $Q_b^{-\delta}$  decays to a positive order. So  $(xx')^{-1}Q_b^{-\delta}$  has leading order greater than  $-1$  at  $rb_0$ ; therefore, the leading order part of  $G_{zf}^0$  at  $rb_0$  is precisely

$$(xx')^{-1}(V^{-1}(-\log \kappa' + \log 2 - \gamma)).$$

But by Bessel function asymptotics,

$$k^{-1}G_{rb_0}^{-1} = V^{-1}(xx')^{-1}K_0(\kappa') \sim (xx')^{-1}V^{-1}(-\log \kappa' + \log 2 - \gamma).$$

Therefore  $G_{zf}^0$  and  $G_{rb_0}^{-1}$  are consistent.

We must also check consistency of  $G_{rb_0}^{-1}$  with  $G_{bf_0}^{-2}$ . Near  $rb_0$ ,

$$k^{-2}G_{bf_0}^{-2} = V^{-1}(xx')^{-1}I_0(\kappa)K_0(\kappa') + (xx')^{-1} \sum_{j=1}^{\infty} \Pi_{E\nu_j} I_{\nu_j}(\kappa)K_{\nu_j}(\kappa').$$

We are only interested in the order  $-1$  part of this term. Since  $x'$  and  $\kappa$  both vanish to first order at  $rb_0$ , all the  $j > 0$  terms have leading order  $-1 + \nu_j$  at  $rb_0$ . Since  $I_0(0) = 1$ , the order  $-1$  part of  $k^{-2}G_{bf_0}^{-2}$  at  $rb_0$  is precisely

$$(xx')^{-1}V^{-1}K_0(\kappa') = k^{-1}(V^{-1}x^{-1}\kappa'K_0(x')) = k^{-1}G_{bf_0}^{-1}.$$

We conclude that  $G_{rb_0}^{-1}$  is consistent with the models at  $zf$  and  $bf_0$ .

We also need to specify some lower order terms at  $bf_0$ ; for this we precisely follow Section 4 of [GH1]. At  $zf$ , they need to match with the asymptotics of  $Q_b^{-\delta}$ , and at  $bf_0$ , they need to match with the higher order Bessel functions. Both of these involve only the nonzero indicial roots, so the terms and arguments are identical to [GH1]. In particular, for any  $0 < \nu_j < 1$  in the indicial set, we let

$$G_{rb_0}^{\nu_j-1} = x^{-1} \frac{\kappa' K_{\nu_j}(\kappa')}{\Gamma(\nu_j)2^{\nu_j-1}} v_{\nu_j}(z, y'),$$

where  $v_{\nu_j}(z, y')$  is in the kernel of  $P_b$  with asymptotic

$$v_{\nu_j}(x, y, y') = (2\nu_j)^{-1} \Pi_{E_j} x^{-\nu_j} + \mathcal{O}(x^{-\nu_j-1} \log x).$$

The function  $v_{\nu_j}$  is there to match with the asymptotics of  $Q_b$  at  $rb_0$ , as in Section 4 of [GH1]. In fact, these models are consistent with our models at  $bf_0$  and at  $zf$  by precisely the same argument as in [GH1]; we will not repeat it here.

**4.5. The final parametrix and resolvent.** We have now constructed models at  $sc$ ,  $bf_0$ ,  $zf$ , and  $rb_0$  which are consistent with each other and also with the diagonal symbol. Moreover, all the models decay to infinite order as we approach  $lb$ ,  $rb$ , or  $bf$ . Therefore, we specify our parametrix  $G(k)$  to be any pseudodifferential operator in  $\Psi_k^{-2, \varepsilon}$  with kernel having the specified diagonal symbol and specified leading-order terms at  $sc$ ,  $bf_0$ ,  $zf$ , and  $rb_0$ . The consistency we checked guarantees that such an operator exists. The behavior of the kernel of  $G(k)$  at  $lb_0$  may be freely chosen as long as the leading-order term is order  $-1$  and it matches with our models at  $zf$  and  $bf_0$ ; a term of order  $-1$  will, however, be required.

Now let  $E(k) = (P + k^2)G(k) - Id$ . Since  $G(k)$  has diagonal symbol equal to the inverse of the symbol of  $P + k^2$ , the Schwarz kernel of  $E(k)$  is smooth on the interior of  $M_{w,sc}^2$ . Moreover, since  $P + k^2$  is a differential operator, the Schwarz kernel of  $E(k)$  is phg conormal on  $M_{w,sc}^2$ .

- At  $lb$ ,  $rb$ , and  $bf$ , the Schwarz kernel of  $G(k)$  vanishes to infinite order along with all derivatives, so the same is true of  $E(k)$ .

- At  $sc$ ,  $G_{sc}^0$  solves the model problem, so  $E(k)$  has positive leading order at  $sc$ .

- At  $bf_0$ ,  $G(k)$  has order  $-2$ , but  $G_{bf_0}^{-2}$  solves the model problem, and moreover  $P + k^2$  vanishes to second order. Therefore  $E(k)$  has positive leading order at  $bf_0$ .

- At  $zf$ ,  $G_{zf}^0$  solves the model problem, so  $E(k)$  has positive leading order.

- At  $lb_0$ ,  $G(k)$  has order  $-1$ . The variables  $k$  and  $x$  both vanish at  $lb_0$ , so  $k^2 G(k)$  has order 1 and  $xG(k)$  has order 0. Since  $P_b$  is a b-differential operator,  $P_b(xG(k))$  also has order 0, and hence  $(P + k^2)G(k) = (xP_b x + k^2)G(k)$  has order 1. Since  $Id$  is supported away from  $lb_0$ ,  $E(k)$  decays to at least order 1 at  $lb_0$ .



- At  $\text{rb}_0$ ,  $G(k)$  has order  $-1$ , but all the terms  $G^{\nu_j-1}$  for  $\nu_j \in [0, 1)$  solve the model problem. Therefore, the error  $E(k)$  has leading order at worst 0.

To summarize, if  $\mathcal{E}$  is the index set for  $E(k)$ , we have shown:

$$\mathcal{E}_{sc} > 0, \mathcal{E}_{zf} > 0, \mathcal{E}_{bf_0} > 0, \mathcal{E}_{lb_0} \geq 1, \mathcal{E}_{rb_0} \geq 0, \mathcal{E}_{lb} = \mathcal{E}_{rb} = \mathcal{E}_{bf} = \emptyset.$$

Now we iterate away the error. By Proposition 20,  $E(k)^2$  vanishes to positive order at all faces of  $M_{k,sc}^2$ ; suppose that the order of vanishing at each face is greater than  $\epsilon > 0$ . Again applying Proposition 20, we see that for each  $N \in \mathbb{N}$ , the order of vanishing of  $E(k)^{2N}$  and  $E(k)^{2N+1}$  at each face of  $M_{k,sc}^2$  is greater than  $N\delta$ . Therefore the Neumann series

$$(Id + E(k))^{-1} = \sum_{i=0}^{\infty} E(k)^i$$

may be summed asymptotically, and the sum defines an element of  $\Psi_k^{-\infty, \hat{\mathcal{E}}}$  for some index family  $\hat{\mathcal{E}}$ .

Finally, let  $R(k) = G(k)(Id + E(k))^{-1}$ ; we see that  $(P + k^2)R(k) = Id$ . Since  $P + k^2$  is invertible for all positive  $k$ , its only right inverse is the resolvent. We conclude that  $R(k)$  is in fact the resolvent, and it is an element of  $\Psi_k^{-\infty, \mathcal{R}}(M, \tilde{\Omega}_b^{1/2})$  for some index family  $\mathcal{R}$ .

In order to prove Theorem 6, we need to perform this construction for any angle  $\theta \in (-\pi, \pi)$ , not just for  $\theta = 0$ . However, as Guillarmou and Hassell claim in [GH1], the construction is essentially unchanged. Indeed, we just use  $e^{i\theta/2}k$  as our boundary defining function for the  $k = 0$  faces instead of  $k$ , and correspondingly change the model at  $sc$  from  $(\Delta_{\mathbb{R}^2} + k^2)^{-1}$  to  $(\Delta_{\mathbb{R}^2} + e^{i\theta}k^2)^{-1}$ . The construction is then precisely analogous to the  $\theta = 0$  case; by construction, the index sets are independent of  $\theta$ . Moreover, by the continuity of the resolvent outside the spectrum (also by construction), all the dependence on  $\theta$  is smooth. This completes the proof.

**4.6. Leading orders of the resolvent.** Since  $R(k) = G(k) - G(k)E(k) + G(k)E(k)^2 - \dots$ , we can obtain some information about the leading orders of  $R(k)$  at each face. For  $n \geq 3$ , it is shown in [GH1] that the leading orders of  $R(k)$  are the same as those of  $G(k)$ ; we claim that the same is true when  $n = 2$ .

When  $n = 2$ ,  $G(k)$  has leading orders  $-1$  at  $\text{lb}_0$  and  $\text{rb}_0$ , order 0 at  $sc$ , order  $-2$  at  $\text{bf}_0$ , and logarithmic growth at  $zf$ .  $E(k)$  has non-negative leading orders at all faces, and it is easy to use Proposition 20 to show that the leading orders of  $G(k)E(k)$  are no worse than those of  $G(k)$ . Similarly, it may be shown that the leading orders of  $G(k)E(k)^l$  are no worse than those of  $G(k)$ . Since  $G(k)$  is fixed and  $Id - E(k) + E(k)^2 - \dots$  is asymptotically summable, the series  $G(k) - G(k)E(k) + \dots$  is also asymptotically summable; therefore, the leading orders of  $R(k)$  are no worse than those of  $G(k)$ . The leading order terms themselves may be affected by  $G(k)E(k)$ , but the orders are not.

To summarize, when  $n = 2$ , the leading orders of the exact resolvent  $R(k)$  are no worse than 0 at  $sc$ ,  $-2$  at  $\text{bf}_0$ , logarithmic at  $zf$ , and  $-1$  at  $\text{lb}_0$  and  $\text{rb}_0$ . When  $n \geq 3$ , the orders are at worst 0 at  $sc$ ,  $-2$  at  $\text{bf}_0$ , 0 at  $zf$ , and  $\frac{n}{2} - 2$  at  $\text{lb}_0$  and  $\text{rb}_0$ , as in [GH1]. However, these orders are for the resolvent acting on  $b$ -half-densities, rather than the more natural scattering half-densities. Switching to scattering half-densities requires an order shift, adding  $n/2$  at each of  $\{x = 0\}$  and  $\{x' = 0\}$ . So we need to add  $n/2$  to the orders at  $\text{lb}_0$  and  $\text{rb}_0$ , and  $n$  at  $\text{bf}_0$ . The order at  $sc$  remains unchanged, because the extra factor of  $\rho_{sc}^{n/2}$  in the definition of the calculus already incorporates the shift. So: viewing the resolvent as a scattering half-density

$|dgdg'|^{1/2}$  acting on scattering half-densities for each  $k$ , or equivalently as a function acting on functions on  $M$  by integration against  $dg$ , it has leading orders given by:

- 0 at sc and  $n - 2$  at  $\text{bf}_0$ ,  $\text{rb}_0$ , and  $\text{lb}_0$ ;
- $r_{zf}$  at  $\text{zf}$ , where  $r_{zf} = 0$  if  $n \geq 3$  and  $r_{zf} = (0, 1)$  (that is, leading order behavior of  $\log \rho_{zf}$ ) if  $n = 2$ .

## APPENDIX A. CONSTRUCTION OF THE SHORT-TIME HEAT KERNEL

Albin has created a framework for the construction of the heat kernel on an asymptotically conic manifold; essentially all of the hard work involved in this construction has already been done in [Alb]. To complete the construction and prove Theorem 3, all we need to do is create an initial parametrix for the heat kernel. This construction is the content of this short appendix and is based on Section 5 of [Alb], in which Albin constructs the heat kernel on an edge manifold.

The space in Theorem 3, which we call  $S$ , is obtained by taking the manifold  $M_{sc}^2 \times [0, T)_t$  and then blowing up the  $t = 0$  diagonal. We call the scattering face of  $M_{sc}^2 \times [0, T)_t$   $\text{sf}$  and call the front face at  $t = 0$   $\text{ff}$ . The heat operator is precisely  $\partial_t + \Delta_M$ . Our goal is to create a parametrix which, to first order, solves the normal equations at  $\text{sf}$  and  $\text{ff}$ .

First analyze the situation at  $\text{sf}$ . As first discussed in [Me3] and elaborated upon in [GH1],  $\text{sf}$  has a Euclidean structure, parametrized by  $y \in N$  and  $t \in [0, T)$ . By the same analysis as in [GH1], the normal operator at  $\text{sf}$  is precisely  $\partial_t + \Delta_{\mathbb{R}^n}$ . We then simply let the model at  $\text{sf}$  be the Euclidean heat kernel,  $H^{\mathbb{R}^n}$ . This is analogous to the construction in Section 5 of [Alb] for the edge setting, in which the model is the heat kernel on hyperbolic space times the heat kernel on the fiber.

The analysis at  $\text{ff}$  is also standard, since  $\text{ff}$  corresponds to the short-time regime on the interior of the manifold, where the heat kernel asymptotics are local. We know that  $d(z, z')/\sqrt{t}$  is a good coordinate along  $\text{ff}$ , zero at the spatial diagonal and increasing to infinity as we approach the original  $t = 0$  face. We therefore let the leading-order model at  $\text{ff}$  be

$$H_{ff} = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(d(z, z'))^2}{4t}}.$$

The choice of model at  $\text{ff}$  is again based on the Euclidean heat kernel, and is precisely the same as the choice of model in the edge setting [Alb].

Each model vanishes to infinite order as we approach all boundary hypersurfaces other than  $\text{sf}$  and  $\text{ff}$ . Moreover, the models are consistent, as the leading orders of each are precisely the Euclidean heat kernel at  $\text{sf} \cap \text{ff}$ . We may therefore pick a pseudodifferential operator whose Schwarz kernel agrees with our models to leading order at  $\text{sf}$  and  $\text{ff}$ , and decays to infinite order at all other boundary faces. In Section 4 of [Alb], Albin proves a composition rule for time-dependent pseudodifferential operators whose kernels are polyhomogeneous conormal on  $S$  - our setting is the 'scattering' setting, which is included in his analysis. We then use this composition rule and an iteration argument, precisely as in Section 5 of [Alb], to construct the heat kernel as a polyhomogeneous conormal distribution on  $S$ . This completes the proof of Theorem 3.

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